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The first-order stochastic dominance ordering of the Singh–Maddala distribution

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Abstract

Given two distributions from the Singh–Maddala family, this paper investigates how to determine whether one distribution first-order stochastically dominates the other. The resulting criteria are also applied to the Dagum type I family of distributions. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

The family of distributions proposed by Singh and Maddala (1976) has been a popular model for describing the distribution of income or consumption expenditure (see McDonald, 1984; Brachmann et al., 1996). The cumulative distribution function (cdf) is given by

$$F(x; a, b, q) = 1 - [1 + (x/b)^a]^{-q}. \tag{1}$$

In empirical applications, the parameters b, a and q are estimated to facilitate intertemporal or international comparisons of income distributions with a view to drawing conclusions about inequality and social welfare. Where inequality is concerned, comparisons are usually made using the Lorenz-ordering. For the Singh–Maddala family, Wilfling and Krämer (1993) have derived necessary and sufficient conditions to determine the Lorenz-ordering of two distributions in terms of the parameters a and q. Since it is mean-free, however, the Lorenz-ordering does not provide an answer to the question which one of two distributions implies higher social welfare. If we focus on the subclass of

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additively separable welfare functions satisfying monotonicity in each individual's income, then a comparison with respect to social welfare will be equivalent to a comparison of the associated cdf's with respect to first-order stochastic dominance (FSD). Moreover, FSD implies higher order stochastic dominance as well as generalized Lorenz-dominance (see Cowell, forthcoming) which is a central concept for comparing income distributions in many fields of economics.[1]

Distribution G is said to (weakly) first-order stochastically dominate H if, and only if, $G(x) \leq H(x)$ for all x (see Cowell, forthcoming). Although some distribution-free tests for FSD have been developed (see Schmid and Trede, 1996, for a recent example), virtually no attempts have been made to address this issue in the context of parametric distributions. This I shall attempt to do for the Singh–Maddala (SM) family.

As in the case of the Lorenz-ordering, the FSD ordering of distributions from the Singh–Maddala family is not complete, i.e. in many cases the cdf's will intersect. Moreover, while for the Lorenz-ordering the scale parameter b plays no role, the FSD ordering will depend on all three parameters, rendering it impossible to give a set of closed form 'if, and only if' conditions.

2. Necessary and sufficient conditions for first-order stochastic dominance

Theorem 1 gives necessary conditions for first-order stochastic dominance.

Theorem 1. *Let F_1 and F_2 be SM distribution functions, with parameters a_i, b_i and q_i* $(i = 1, 2)$, *respectively. If F_1 first-order stochastically dominates F_2, then*

(a) $a_1 \geq a_2$ *and*
(b) $a_1 q_1 \leq a_2 q_2$.

Proof. Define the family of strictly increasing functions $u_t(x) = x^t/t$, $t \neq 0$ and the corresponding family of additively separable social welfare functions $W_t(F) = \int u_t(x)\, \mathrm{d}F(x) = \mu_t(F)/t$, where $\mu_t(F)$ denotes the tth moment associated with F. We further need the following representation of the tth moment of the SM family obtained by McDonald (1984):

$$\mu_t(F) = b^t \Gamma(1 + t/a)\Gamma(q - t/a)/\Gamma(q) \tag{2}$$

where $\Gamma(\cdot)$ denotes the complete gamma function.

(a) Assume that $a_1 < a_2$ and let t approach $-a_1$ from above. In this case, $(1 + t/a_1)$ will approach zero. Inspection of (2) and recalling that $\lim_{z\to 0} \Gamma(z) = \infty$ reveals that this implies that $W_t(F_1)$ will approach minus infinity, while $W_t(F_2)$ will approach a finite negative number. Thus for some $t' > -a_1$ we have $W_{t'}(F_1) < W_{t'}(F_2)$. Since $F_1(x) \leq F_2(x)$ for all x implies $W_t(F_1) \geq W_t(F_2)$ for all t (see Saposnik, 1981), $W_{t'}(F_1) < W_{t'}(F_2)$ contradicts $F_1 \leq F_2$.

(b) Assume that $a_1 q_1 > a_2 q_2$ and let t approach $a_2 q_2$ from below. Now the term $\Gamma(q_2 - t/a_2)$ and thus $W_t(F_2)$ will approach plus infinity, while $W_t(F_1)$ will approach a finite positive number. Thus for some $t^* < a_2 q_2$ we have $W_{t^*}(F_1) < W_{t^*}(F_2)$, which contradicts $F_1 \leq F_2$. QED

[1] See Lambert (1989) for an application to public finance.

It is interesting to note that the necessary conditions given in Theorem 1 are in direct contrast to those for Lorenz-dominance obtained by Wilfling and Krämer (1993) who showed that F_1 Lorenz-dominates F_2 if, and only if, $a_1 \geq a_2$ and $a_1 q_1 \geq a_2 q_2$. Thus, if two distributions can be Lorenz-ordered, there will be no ordering according to FSD and vice versa. This is a serious drawback of the SM family since, in general, a distribution G can first-order and Lorenz-dominate a distribution H at the same time.

We state an inequality for sums introduced by Pringsheim (1902a,b, see also Hardy et al., 1952, Theorem 19) that will be needed for the proof of Theorem 2:

Lemma 1. *For positive r, p and c_k, $k = 1, \ldots, n$, $(\sum_{k=1}^{n} c_k^r)^{1/r} \leq (\sum_{k=1}^{n} c_k^p)^{1/p}$ if, and only if, $r \geq p$.*

A set of sufficient conditions for first-order stochastic dominance is set out in:

Theorem 2. *If $a_1 \geq a_2$, $a_1 q_1 \leq a_2 q_2$ and $b_1 \geq b_2$, then F_1 first-order stochastically dominates F_2.*

Proof. For the SM family $F_1 \leq F_2$ is equivalent to

$$(1 + (x/b_1)^{a_1})^{q_1} \leq (1 + (x/b_2)^{a_2})^{q_2}. \tag{3}$$

Since $(1 + (x/b)^a)^q$ is decreasing in b and increasing in q and, by hypothesis, $b_1 \geq b_2$ and $a_1 q_1 \leq a_2 q_2$, we have $(1 + (x/b_1)^{a_1})^{q_1} \leq (1 + (x/b_2)^{a_1})^{a_2/a_1 q_2}$. It thus suffices to show that $(1 + (x/b_2)^{a_1})^{a_2/a_1 q_2} \leq (1 + (x/b_2)^{a_2})^{q_2}$ which, after a change of variable, is equivalent to $(1 + z^{a_1})^{1/a_1} \leq (1 + z^{a_2})^{1/a_2}$ for all positive z. That this holds for all $0 < a_2 \leq a_1$ follows at once from Lemma 1. QED

The next theorem covers the special cases where either (a) or (b) of Theorem 1 holds with equality and, as a third case, $b_1 \geq b_2$:

Theorem 3.

(a) Suppose $a_1 = a_2 = a$. Then F_1 first-order stochastically dominates F_2 if, and only if, $q_1 \leq q_2$ and $b_1/b_2 \geq (q_1/q_2)^{1/a}$.
(b) Suppose $a_1 q_1 = a_2 q_2$. Then F_1 first-order stochastically dominates F_2 if, and only if, $a_1 \geq a_2$ and $b_1 \geq b_2$.
(c) Suppose $b_1 \geq b_2$. Then F_1 first-order stochastically dominates F_2 if, and only if, $a_1 \geq a_2$ and $a_1 q_1 \leq a_2 q_2$.

Proof. (a) In this case, after a change of variable, $F_1 \leq F_2$ is equivalent to $(1 + z)^{q_1/q_2} \leq 1 + (b_1/b_2)^a z$ for all positive z. Clearly, this inequality will hold for all z if, and only if, q_1/q_2 is not bigger than unity and, for $z = 0$, the derivative of the left hand side is not bigger than the derivative of the right hand side. This latter condition is equivalent to $b_1/b_2 \geq (q_1/q_2)^{1/a}$.

(b) Necessity of $a_1 \geq a_2$ follows from Theorem 1. For the necessity of $b_1 \geq b_2$, rewrite $F_1 \leq F_2$ as $b_2/b_1 \leq (1 + (b_2/x)^{a_2})^{1/a_2}/(1 + (b_1/x)^{a_1})^{1/a_1}$. The right hand side of this inequality approaches unity for large x. Thus $b_2 > b_1$ contradicts $F_1 \leq F_2$. Sufficiency follows from Theorem 2.

(c) Necessity follows from Theorem 1, sufficiency from Theorem 2. QED

In many cases, Theorems 1, 2 and 3 will not suffice to determine whether the cdf's of two

Table 1
Values of $v(a_1/a_2,\ q_2/q_1)$

	$q_2/q_1 \rightarrow$	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3
$a_1/a_2\downarrow$	1	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90	2.00	2.10	2.20	2.30	2.40	2.50	2.60	2.70	2.80	2.90	3.00
	1.1		1.00	1.25	1.41	1.56	1.70	1.84	1.97	2.11	2.24	2.37	2.50	2.63	2.76	2.89	3.02	3.15	3.27	3.40	3.53	3.66
	1.2			1.00	1.28	1.47	1.64	1.80	1.96	2.11	2.26	2.41	2.55	2.70	2.84	2.99	3.13	3.27	3.41	3.55	3.70	3.84
	1.3				1.00	1.28	1.49	1.67	1.84	2.01	2.17	2.32	2.48	2.63	2.78	2.93	3.08	3.23	3.38	3.53	3.68	3.82
	1.4					1.00	1.28	1.49	1.67	1.85	2.01	2.18	2.34	2.49	2.65	2.80	2.96	3.11	3.26	3.41	3.56	3.71
	1.5						1.00	1.27	1.47	1.66	1.83	2.00	2.16	2.32	2.48	2.63	2.79	2.94	3.09	3.24	3.39	3.54
	1.6							1.00	1.26	1.46	1.64	1.81	1.98	2.14	2.29	2.45	2.60	2.75	2.90	3.05	3.20	3.35
	1.7								1.00	1.25	1.44	1.62	1.79	1.95	2.10	2.26	2.41	2.56	2.70	2.85	3.00	3.14
	1.8									1.00	1.24	1.43	1.60	1.76	1.91	2.07	2.22	2.36	2.51	2.65	2.79	2.94
	1.9										1.00	1.23	1.41	1.57	1.73	1.88	2.03	2.17	2.32	2.46	2.60	2.74
	2											1.00	1.22	1.39	1.55	1.70	1.85	1.99	2.13	2.27	2.41	2.54
	2.1												1.00	1.21	1.38	1.53	1.68	1.82	1.96	2.09	2.23	2.36
	2.2													1.00	1.21	1.37	1.51	1.65	1.79	1.92	2.06	2.18
	2.3														1.00	1.20	1.35	1.50	1.63	1.76	1.89	2.02
	2.4															1.00	1.19	1.34	1.48	1.61	1.74	1.86
	2.5																1.00	1.18	1.33	1.46	1.59	1.71
	2.6																	1.00	1.18	1.32	1.45	1.57
	2.7																		1.00	1.17	1.31	1.43
	2.8																			1.00	1.17	1.30
	2.9																				1.00	1.16
	3																					1.00

distributions under comparison intersect or not, namely when conditions (a) and (b) of Theorem 1 both hold with strict inequality together with $b_2 > b_1$. Therefore I provide a table that, together with Theorem 1, fills this gap. First note that $F_1 \leq F_2$ is equivalent to

$$(b_2/b_1)^{a_2} \leq z/\left((1+z^{a_1/a_2})^{q_1/q_2} - 1\right) \tag{4}$$

for all positive z. Further, the necessary conditions of Theorem 1 can be written as

$$1 \leq a_1/a_2 \leq q_2/q_1. \tag{5}$$

Table 1 therefore reports the minimum of the r.h.s. of (4) with respect to z, $v(a_1/a_2,\ q_2/q_1)$, for pairs of a_1/a_2 and q_2/q_1 over a range that is sufficient for most applications and that satisfies (5). Thus, F_1 first-order stochastically dominates F_2 if, and only if, (a) and (b) of Theorem 1 hold and $(b_2/b_1)^{a_2} \leq v(a_1/a_2,\ q_2/q_1)$.

3. The dagum type I family

The cdf of the Dagum type I model (Dagum, 1977) is given by

$$G(x; \tilde{a}, \tilde{b}, \tilde{q}) = \left(1 + (\tilde{b}/x)^{\tilde{a}}\right)^{-\tilde{q}}.$$

A comparison of this with the SM family can be found in Kleiber (1996), who also obtained necessary and sufficient conditions for Lorenz-dominance.

Rearranging (3) together with a change of variable yields the following lemma.

Lemma 2. *Let G_1 and G_2 be Dagum type I distribution functions with parameters $\tilde{a}_i$, $\tilde{b}_i$ and $\tilde{q}_i$* ($i = 1,2$), *respectively, and define $F(x; a, b, q)$ as in* (1). *Then G_1 first-order stochastically dominates G_2 if, and only if, $F(x; \tilde{a}_2, 1/\tilde{b}_2, \tilde{q}_2)$ first-order stochastically dominates $F(x; \tilde{a}_1, 1/\tilde{b}_1, \tilde{q}_1)$.*

Thus all results obtained for the SM family in Section 2 can be applied to the Dagum type I family if we replace a_1, b_1 and q_1 by $\tilde{a}_2$, $1/\tilde{b}_2$ and $\tilde{q}_2$, respectively, and a_2, b_2 and q_2 correspondingly.

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