A Model of Money Non-Neutrality under Heterogeneous Beliefs

Jörg Rieger†

November 2012

Abstract

This paper studies the effects of money in an infinite-horizon general equilibrium model with cash-in-advance constraints. The central bank injects money into the economy in exchange of the endowment of the consumption good. Agents hold money for store of value as well as for transactions purposes and form heterogeneous beliefs about future variables. The class of beliefs is required to be compatible with the data or empirical frequencies of all observables. The concept of rational beliefs is based on the fact that many stochastically stable processes (e.g. regime switching models) generate the same data, whose long-run frequencies are the stationary measure. Heterogeneity in beliefs is given by a market state of beliefs, which is an anonymous distribution of beliefs of the individual agents and generates correlation between the beliefs of individual agents. We show that in contrast to a stationary equilibrium with homogeneous rational expectations, a heterogeneity in beliefs produces a real effect of money on the fluctuations of economic variables, such as prices and consumption. Moreover, macroeconomic volatility, as measured by the above parameters, tends to be amplified in the presence of heterogeneous beliefs.

*I thank Hiro Nakata, Maurizio Motolese and Jerry Coakley for and Participants at the Polhia Conference in Milan 2011 for helpful comments. All errors are mine.
†Address for Correspondence: Alfred Weber-Institut, University of Heidelberg, Bergheimer Str. 58, 69115 Heidelberg. E-mail: joerg.rieger@awi.uni-heidelberg.de. Tel.: +49 (0)6221 542935
1 Introduction

How does money affect allocation and prices in equilibrium? While empirical results provide evidence that money is not neutral (Mishkin (1982)), in models with rational expectations, no asymmetric information and no trading frictions money is neutral (Lucas (1972), Sargent and Wallace (1975)). This indicates several ways to achieve the non-neutrality of money. First, one can consider transactions costs (Rotemberg (1984), Grossmann and Weiss (1983), Williamson (2008)). Second, firms cannot adjust prices every period (Calvo (1983)). Third, by using the insight of Lucas (1972), asymmetric information is also a possibility (Ui (2003), Woodford (2003)) to achieve non-neutrality. Another way is to allow for heterogeneous beliefs (Motolese (2001, 2003)).

As there is considerable evidence that the households in the economy do have heterogeneous beliefs (see e.g. Pesaran and Weale (2006), Mankiw et al. (2003)), the focus of this paper will be on heterogeneous beliefs as the cause for the non-neutrality of money. To model heterogeneous beliefs, we use a special class of heterogeneous beliefs, called rational beliefs, as introduced by Kurz (1994). Under the rational beliefs principle, the households’ beliefs cannot be arbitrary, but have to satisfy a rationality condition. The rationality condition only demands that the households’ beliefs have to be compatible with the empirical distribution, which does not rule out the possibility that households make incorrect forecasts.

Because the households make incorrect forecasts, the volatility of endogenous variables, such as prices, is greater in an rational belief equilibrium than in an rational expectations equilibrium. This has been used to explain the excess volatility of asset prices (Kurz and Beltratti (1997), Kurz et al. (2005a)), the trading volume of assets (Wu and Guo (2003)). And it has also been used to calibrate a simple model to match the moments of US-Macroeconomic variables (Kurz et al. (2005b)). This lets the question open, how does monetary policy affects the excess volatility in an economy with heterogeneous beliefs? And how do beliefs affect the
efficiency of said monetary policy?

Without any further assumptions, money has no role in equilibrium beyond setting the prices. To facilitate the transactions role of money, one has to model how the households trade. A common way is to use a cash-in-advance constraint. Building on the idea by Clower (1967) that 'goods buy money, money buys goods, but goods do not buy goods' the cash-in-advance constraint restricts the households access to different markets such as the asset market or the goods market.

In addition to the transactions role of money, money can also be part of the portfolio decisions of the households. If there are other risk free assets in the economy, e.g. bonds, the returns of these other risk free assets will dominate the return of money. Which means that rational agents will not hold money as a store of value. Thus, for money to be held as a store of value, we have to assume that money is the only risk free asset in the economy. This approach is not new and has been used in the literature (see e.g. Samuelson (1958), Bewley (1983), Kiyotaki and Moore (2008)).

We are studying an economy with two infinite-lived households, who hold rational beliefs. The households can invest in a risky asset, which is in zero net-supply and which pays nominal dividends. They also hold money as a store of value. To facilitate the transactions role of money, we use a cash-in-advance constraint due to Magill and Quinzii (1992). Furthermore, the central bank supplies the economy with a fixed amount of money every period.

We follow Kurz et al. (2005b) and consider beliefs as part of the economic primitives. While the individual beliefs cannot be observed, the anonymous distribution of the beliefs of the households, called the market state of belief, is observable. Extending the state-space with the market-state of belief causes the excess volatility in the economy for two reasons. First, prices in the economy depend on the market-state of belief, and thus on the beliefs and decisions of the households. Second, to forecast prices, the households have to forecast future market state of beliefs. This adds additional uncertainty to the economy, since the households do make different forecasts.

We then study numerically the properties of the stationary equilibrium using a second-order perturbation method (Jin and Judd (2002), Schmitt-Grohe and Uribe (2004)). The numerical results indicate that under rational beliefs increasing the
money supply can decrease the volatility of consumption, because the households face now higher risk about the decisions of the other households. Thus, the households have less incentives to trade assets and the trading volume of the asset goes down. As the trading volume of the asset goes down, the volatility of consumption declines.

The paper is structured as follows. We discuss the structure of the economy in section 2. In section 3, we define rational beliefs and explain the role of the individual and market state of beliefs. We then discuss the results of the numerical simulations in section 4. And section 5 concludes the paper.

2 The Model

In this section we discuss the structure of the economy. We first discuss the real and the financial side of the economy, in particular the households preferences, the endowment process and the dividend process of the risky asset. We then move on to discuss the cash-in-advance constraint on the individual and aggregate level. This section concludes with the market clearing conditions.

2.1 Households, Endowments and Dividends

The economy consists of two infinite-lived households, denoted by $h = 1, 2$. The endowment of the households with the perishable consumption good is stochastic and follows an AR(1) process:

$$
\omega^h_{t+1} = (1 - \lambda^h)\bar{\omega}^h + \lambda^h \omega^h_t + \epsilon^h_{t+1}, \quad \epsilon^h_{t+1} \sim i.i.d. N(0, \sigma^h).
$$

(1)

With $\bar{\omega}^h$ as the average endowment of household $h = 1, 2$, $\sigma^h$ as the standard deviation of the shock and $\lambda^h$ as the parameter of the autoregressive process. The correlation of the shocks across households is $\sigma_{\omega^1, \omega^2}$.

The preferences of a household $h = 1, 2$ can be represented by a CARA-utility function:

$$
U^h = \sum_{s=0}^{\infty} \beta^s \frac{1}{\gamma^h} E_{Q^h_t} \{ \exp \{-\gamma^h c^h_{t+s}\} | I_t \},
$$

(2)
with \( Q^h_t \) as the beliefs of household \( h \) formed at time \( t \). The beliefs \( Q^h_t \) are time-varying and how the households form their beliefs is explained in section 3. The households’ information set at time \( t \) is denoted by \( I_t \) and all information is public. The discount factor is denoted by \( \beta \) and is equal for all households. The risk-aversion of a household is denoted by \( \gamma^h \).

The households can save by investing into a risky asset and there are no explicit short-sale constraints on the asset holdings for the households. The only assumption made is that there are no arbitrage opportunities. They also can hold money as a store of value.

The risky asset is long-lived and in zero net-supply. It also pays nominal dividends. In particular, we assume that the dividends follows an AR(1) process:

\[
v_{t+1} = (1 - \lambda_v) \bar{v} + \lambda_v v_t + \epsilon^v_t, \quad \epsilon^v_t \sim i.i.d. N(0, \sigma_v).
\] (3)

With \( \bar{v} \) as the average and \( \lambda_v \) as the parameter of the dividend process and \( \sigma_v \) as the standard deviation of the dividend shock. The correlation between the endowment shock of household \( h \) and the dividend shock is denoted by \( \sigma_{v\omega^h} \).

The second asset is money, which also serves as a risk-less asset and as a means of payment. The money supply in the economy is fixed.

In the initial period \( t = 0 \), the households asset and money holdings are zero.

### 2.2 The Individuals’ Optimization Problem

Every period is divided into further three sub periods. In the first sub period, agents are forced to sell their endowment and receive \( m^h_t \):

\[
m^h_t = p_t \omega^h_t.
\] (4)

In the second sub period, the asset market opens. The households decide now on their savings. They decide now how much money they keep as a store of value and how much to invest in the risky asset. Furthermore, they receive now the dividends from their asset holdings.

\[
\hat{m}^h_{t+2} = m^h_t - q_t (\theta^h_t - \theta^h_{t-1}) + v_t \theta^h_{t-1} - \theta^h_{m,t} + \theta^h_{m,t-1},
\]
where $\theta_h^t$ denote the asset holdings of the agent for the risky asset and $\theta_{m, t}$ the amount of money the households uses as the store of value.

In the third sub period the household spent their money on the consumption good.

$$p_t^B c_t^h = m_t^h - q_t(\theta^h_t - \theta^h_{t-1}) + v_t \theta^h_{t-1} - \theta^h_{m, t} + \theta^h_{m, t-1},$$  \hspace{1cm} (5)

with $p_t^B$ as the buying price of the consumption good in the third subperiod. Combining now equations (4) and (5) yields the following budget constraint for the household

$$p_t^B c_t^h = p_t \omega^h_t - q_t(\theta^h_t - \theta^h_{t-1}) + v_t \theta^h_{t-1} - (\theta^h_{m, t} - \theta^h_{m, t-1}).$$

Therefore, the households’ optimization problem at time $t$ is now:

$$\max \left\{ \gamma_t c_t^h + s, \theta_t^h + s, \theta_{m, t} + s \right\}$$

$$\text{s.t. } p_t^B c_t^h = p_t \omega^h_t - q_t(\theta^h_t - \theta^h_{t-1}) + v_t \theta^h_{t-1} - (\theta^h_{m, t} - \theta^h_{m, t-1}).$$

The Euler equations for this optimization problem are now given by

$$\exp\{-\gamma h c_t^h\} \frac{q_t^B}{p_t^B} = \beta E_{Q_t}^h [\exp\{-\gamma h c_{t+1}^h\} \frac{q_{t+1} + v_{t+1}}{p_{t+1}^B} | I_t],$$

$$\exp\{-\gamma h c_t^h\} \frac{1}{p_t^B} = \beta E_{Q_t}^h [\exp\{-\gamma h c_{t+1}^h\} \frac{1}{p_{t+1}^B} | I_t].$$

### 2.3 Aggregate Economy

As stated, all the agents will sell their endowments, hence on the aggregate level we have

$$p_t \sum_{h=1}^{2} \omega_t^h = M_1.$$

Therefore, in the first sub period the central bank injects $M_1$ units of money. Given the aggregate endowment, $\sum_{h=1}^{2} \omega_t^h$, we have the selling price $p_t$ for the consumption good.
In the second sub period, the young agents decide how much to spend on the risky asset and the old agents receive the dividends from asset holdings. But as the aggregate supply of the risky asset is zero, these decisions do not affect the aggregate money supply in the economy.

In the third sub period, they decide how much to spend on the consumption good, and how much money to hold as a store of value. Therefore, in the third sub period, the aggregate money $M_{3,t}$ left for consumption is now $M_{3,t} = M_1 - \sum_{h=1}^{2}(\theta_{m,t}^h - \theta_{m,t-1}^h)$. Thus, the buying price $p_t^B$ is now given by:

$$p_t^B \sum_{h=1}^{2} c_t^h = M_{3,t}.$$

### 2.4 The Market Clearing Conditions

Finally, we define the market clearing conditions for the economy.

1. The goods market is cleared:

$$\sum_{h=1}^{2} c_t^h = \sum_{h=1}^{2} \omega_t^h;$$

2. the market for the risky asset is cleared:

$$\sum_{h=1}^{2} \theta_t^h = 0;$$

3. the money market in the first sub period is cleared:

$$p_t \sum_{h=1}^{2} \omega_t^h = M_1;$$

4. the money market in the third sub period is cleared:

$$\sum_{h=1}^{2} \theta_{m,t}^h \leq M_1.$$
3 Rational Beliefs

While the households may not believe that a fixed stationary model captures the empirical distribution implied by (1) and (3), they form different belief to forecast prices and other economic variables. Instead of describing how the households form their beliefs, we follow Kurz et al. (2005b) and assume that the households’ beliefs are part of the stochastic primitives of the economy.

3.1 Market and Individual State of Beliefs

In the first step, we introduce the individual state of belief $g^h_t$. The individual state of belief represents the conditional probabilities of the transition for the economy’s state variables. We assume that the individual state of belief is governed by a process of the form:

$$g^h_t = \lambda_z g^h_t + \lambda_v (v_t - \bar{v}) + \lambda_{\omega^1} (\omega^1 - \bar{\omega}^1) + \lambda_{\omega^2} (\omega^2 - \bar{\omega}^2) + \tilde{\epsilon}_{t+1}^h, \quad \tilde{\epsilon}_{t+1}^h \sim N(0, \tilde{\sigma}_{\epsilon^h}^2). \quad (6)$$

Across households, the shocks $\tilde{\epsilon}_{t+1}^h$ can be correlated to reflect communication among households (Nakata (2007)). In addition, the households only observe their own state of belief, but not the other households state of belief.

In equilibrium, the decisions of the households depend on their state of belief $g^h_t$. Which in turn implies that prices in equilibrium also depend on the beliefs $g_t = (g^1_t, g^2_t)$. However, if a household would recognize this fact, this would give him some market power. To rule out this possibility, we assume the principle of anonymity. The principle of anonymity states that households think that they do not affect the endogenous variables, similar to the price-taking behaviour in competitive markets.

We define now the market state of belief as $z_t = (z^1_t, z^2_t)$ and apply the internal consistency condition $z_t = g_t$. The internal consistency condition is not recognized by the households, thus they see the market state of belief as a variable they cannot influence. In addition, the internal consistency condition implies that the prices in the economy are now a function of $z_t$ instead of $g_t$. While the individual beliefs are not observed by others, we assume that the market state of belief, i.e. the distribution of individual beliefs, is observable by the households. The anonymity
condition also implies that households do not know the identity $h$ of $z^h_t$.

Extending the state-space of the economy with the market-state of belief has two consequences.

First, by adding the market states of belief to the state-space we have new uncertainty. As the prices in equilibrium depend on the market state of belief, a map for the prices can be defined as

$$
\begin{bmatrix}
  p_t \\
  p^B_t \\
  q_t
\end{bmatrix} = \Xi(v^t_t, \omega^1_t, \omega^2_t, z^1_t, z^2_t).
$$

(7)

This gives us now the second implication of the extended state-space. To forecast the prices, households also have to forecast the actions of the other households. While all households use (7) to forecast prices, households will make different forecasts, because the forecast of $(v^t_{t+1}, \omega^1_{t+1}, \omega^2_{t+1}, z^1_{t+1}, z^2_{t+1})$ will depend on his own state of belief $g^h_t$.

We assume that the empirical-distribution of the state-variables is an AR-process of the form

\begin{align*}
  v^t_{t+1} &= (1 - \lambda_v)\bar{v} + \lambda_v v^t_t + \epsilon^v_{t+1}, \tag{8a} \\
  \omega^1_{t+1} &= (1 - \lambda_{\omega^1})\bar{\omega}^1 + \lambda_{\omega^1} \omega^1_t + \epsilon^\omega^1_{t+1}, \tag{8b} \\
  \omega^2_{t+1} &= (1 - \lambda_{\omega^2})\bar{\omega}^2 + \lambda_{\omega^2} \omega^2_t + \epsilon^\omega^2_{t+1}, \tag{8c} \\
  z^1_{t+1} &= \lambda_{z^1} z^1_t + \lambda^1_{v^t} (v^t_t - \bar{v}) + \lambda^1_{z^1, \omega^1^t} (\omega^1_t - \bar{\omega}^1) + \lambda^1_{z^1, \omega^2^t} (\omega^2_t - \bar{\omega}^2) + \epsilon^{z^1}_{t+1}, \tag{8d} \\
  z^2_{t+1} &= \lambda_{z^2} z^2_t + \lambda_{v^t}^2 (v^t_t - \bar{v}) + \lambda^{z^2}_{z^1} (\omega^1_t - \bar{\omega}^1) + \lambda^{z^2}_{z^2} (\omega^2_t - \bar{\omega}^2) + \epsilon^{z^2}_{t+1}. \tag{8e}
\end{align*}

\[
\begin{pmatrix}
  \epsilon^v_t \\
  \epsilon^\omega^1_t \\
  \epsilon^\omega^2_t \\
  \epsilon^1_t \\
  \epsilon^2_t
\end{pmatrix} \sim N\left(\begin{pmatrix}
  0 & \sigma^2_v & \sigma_{v^t} & \sigma_{v^t, \omega^1} & 0 & 0 \\
  0 & \sigma_{v^t} & \sigma^2_{\omega^1} & \sigma_{v^t, \omega^1} & 0 & 0 \\
  0 & \sigma_{v^t, \omega^1} & \sigma^2_{\omega^1, \omega^2} & \sigma^2_{\omega^2} & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & \sigma_{z^1, z^2} \\
  0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}\right).
\]
We can write the previous system of equation in a compact matrix by letting $A$ be the constants and $B$ be the parameters. Furthermore, let $x_t = (v_t, \omega^1_t, \omega^2_t, z^1_t, z^2_t)$ and $\epsilon_t = (\epsilon_t^v, \epsilon_t^\omega^1, \epsilon_t^\omega^2, \epsilon_t^z^1, \epsilon_t^z^2)$. Thus, we have

$$x_{t+1} = A + B x_t + \epsilon_{t+1}, \quad \epsilon_{t+1} \sim N(0, \Sigma). \quad (10)$$

With $\Sigma$ as the Variance-Covariance Matrix. Let $\Gamma$ be the stationary measure implied by (8a)-(8e). Let $V$ be the unconditional covariance matrix of $x$ defined by $V = E[xx']$ and $V$ satisfies

$$V = BB' + \Sigma. \quad (11)$$

### 3.2 The Perception Model

We first define the perception model of a household. The perception model are the transition functions representing the households’ conditional probability belief. Thus, the perception model is used to determine the households’ forecast. The beliefs of the households can deviate from the empirical forecast. This deviation of the forecasts can be interpreted as over-confidence of the households.

Let $x^h_{t+1} = (v^h_{t+1}, \omega^1_{t+1}, \omega^2_{t+1}, z^1_{t+1}, z^2_{t+2})$ be the date $t + 1$ variables as perceived by household $h$ and $\Gamma$ be the stationary measure as implied by (8a)-(8e). We also have $\Psi_{t+1}(g^h_t)$ as a five-dimensional vector of date $t + 1$ random variables conditional upon $g^h_t$.

**Definition 1.** A perception model in the economy under study has the general form

$$x^h_{t+1} = A + B x_t + \Psi_{t+1}(g^h_t) \text{ together with (6).} \quad (12)$$

Since $E_{\Gamma}[x_{t+1}|I_t] = A + B x_t$, we can write (12) as follows

$$x^h_{t+1} - E_{\Gamma}[x_{t+1}|I_t] = \Psi_{t+1}(g^h_t). \quad (13)$$

In the next step, we assume that $\Psi_{t+1}(g^h_t)$ is a function of one random variable $n^h_{t+1}(g^h_t)$. We also assume that the components of the function $\Psi_{t+1}(g^h_t)$ take the
following form

\[
\Psi_{t+1}(g^h_t) = 
\begin{pmatrix}
\lambda^v_y \eta^h_{t+1}(g^h_t) + \tilde{\epsilon}^h_{t+1} \\
\lambda^w_y \eta^h_{t+1}(g^h_t) + \tilde{\epsilon}^h_{t+1} \\
\lambda^z_y \eta^h_{t+1}(g^h_t) + \tilde{\epsilon}^h_{t+1} \\
\lambda^x_y \eta^h_{t+1}(g^h_t) + \tilde{\epsilon}^h_{t+1} \\
\lambda^z_y \eta^h_{t+1}(g^h_t) + \tilde{\epsilon}^h_{t+1} \\
\lambda^x_y \eta^h_{t+1}(g^h_t) + \tilde{\epsilon}^h_{t+1}
\end{pmatrix}
\]

With \( \tilde{\epsilon}^h_{t+1} = (\tilde{\epsilon}_{t+1}^v, \tilde{\epsilon}_{t+1}^w, \tilde{\epsilon}_{t+1}^z, \tilde{\epsilon}_{t+1}^x) \). Thus, we can define now the perception model of a household \( h \) as follows

\[
\begin{align*}
\nu^h_{t+1} &= \bar{v} + \lambda_v \Delta v_t + \lambda^v_y \eta^h_{t+1}(g^h_t) + \tilde{\epsilon}^h_{t+1}, \\
\omega^h_{t+1} &= \bar{\omega}^1 + \lambda_{\omega^1} \Delta \omega^1_t + \lambda^\omega_y \eta^h_{t+1}(g^h_t) + \tilde{\epsilon}^\omega_{t+1}, \\
\omega^h_{t+1} &= \bar{\omega}^2 + \lambda_{\omega^2} \Delta \omega^2_t + \lambda^\omega_y \eta^h_{t+1}(g^h_t) + \tilde{\epsilon}^\omega_{t+1}, \\
z^h_{t+1} &= \lambda_z z^1_t + \lambda_z \Delta v_t + \lambda_{\omega^1} \Delta \omega^1_t + \lambda_{\omega^2} \Delta \omega^2_t + \tilde{\epsilon}^z_{t+1}, \\
z^h_{t+1} &= \lambda_z z^2_t + \lambda_z \Delta v_t + \lambda_{\omega^1} \Delta \omega^1_t + \lambda_{\omega^2} \Delta \omega^2_t + \tilde{\epsilon}^z_{t+1}, \\
g^h_{t+1} &= \lambda_z g^h_t + \lambda_z \Delta v_t + \lambda_{\omega^1} \Delta \omega^1_t + \lambda_{\omega^2} \Delta \omega^2_t + \tilde{\epsilon}^g_{t+1},
\end{align*}
\]

with \( \Delta v_t = v_t - \bar{v}, \Delta \omega^h = \omega^h_t - \bar{\omega}^h \) and \( \tilde{\epsilon}^h_{t+1} = (\tilde{\epsilon}_{t+1}^v, \tilde{\epsilon}_{t+1}^w, \tilde{\epsilon}_{t+1}^z, \tilde{\epsilon}_{t+1}^x) \) i.i.d. Normal with mean zero and covariance matrix \( \Omega^h \) of the form

\[
\Omega^h = 
\begin{pmatrix}
\Omega_{\epsilon \epsilon} & \Omega_{\epsilon y^h} \\
\Omega_{y^h \epsilon} & \sigma_y^2
\end{pmatrix}
\]

where \( \Omega_{y^h} = [\text{cov}(\tilde{\epsilon}_{t+1}^v, \tilde{\epsilon}_{t+1}^h), \text{cov}(\tilde{\epsilon}_{t+1}^w, \tilde{\epsilon}_{t+1}^h), \text{cov}(\tilde{\epsilon}_{t+1}^z, \tilde{\epsilon}_{t+1}^h), \text{cov}(\tilde{\epsilon}_{t+1}^x, \tilde{\epsilon}_{t+1}^h), \text{cov}(\tilde{\epsilon}_{t+1}^{2v}, \tilde{\epsilon}_{t+1}^h)] \).

### 3.3 Modelling the Random Vector \( \Psi_{t+1}(g^h_t) \)

The overconfidence of the households is reflected by the intensity of fat tails in the households’ beliefs. These fat tails are modelled with the random variable \( \eta^h_{t+1}(g^h_t) \).
The random variable $\eta_{h+1}(g^h_t)$ is defined by its density via a step-function:

$$p(\eta_{h+1}|g^h_t) = \begin{cases} 
\psi_1(g^h_t)\Phi(\eta_{h+1}) & \text{if } \eta_{h+1} \geq 0 \\
\psi_2(g^h_t)\Phi(\eta_{h+1}) & \text{if } \eta_{h+1} < 0 
\end{cases}$$

with $\eta^h$ and $\tilde{\epsilon}_{t+1}^h$ independent. And the function $\Phi(\eta) = [1/\sqrt{2\pi}]e^{-\eta^2/2}$. The functions $\psi_1$ and $\psi_2$ are defined by a logistic function with a single parameter $b$.

$$\psi(g^h_t) = \frac{1}{1 + e^{by^h}}, \quad b < 0, \quad G \equiv E_g[\psi(g^h)]$$

$$\psi_1(g^h) = \frac{\psi(g^h)}{G}, \quad \psi_2(g^h) = 2 - \psi_1(g^h)$$

The parameter $b$ measures the intensity of fat tails in the beliefs, the construction is explained in Kurz et al. (2005b).

We can now define what $g^h_t$ means in terms of overconfidence:

**Definition 2.** Let $Q^h$ be the probability belief of household $h$. Then $g^h_t$ is household $h$'s state of overconfidence in abnormally high dividends, if $E_{Q^h} [v^h_{t+1}|g^h_t, I_t] > E_{\Gamma} [v^h_{t+1}|I_t]$; overconfidence in abnormally low dividends if $E_{Q^h} [v^h_{t+1}|g^h_t, I_t] < E_{\Gamma} [v^h_{t+1}|I_t]$.

### 3.4 Restriction on Beliefs under the Rational Beliefs Principle

The following definition provides the definition of Rational Beliefs due to Kurz (1994).

**Definition 3.** A perception model as defined in (14a)-(14f) and (15) is a rational belief if the household’s model $x^h_{t+1} = A + Bx_t + \Psi_{t+1}(g^h_t)$ together with (6) has the same empirical distribution as $x_{t+1} = A + Bx_t + \epsilon_{t+1}$.

As a rational beliefs model cannot be rejected by the data, it has to match all moments of the observables. This implies that $\Psi_{t+1}$ must have the same distribution as $\epsilon_{t+1}$. While the rational beliefs principle allows the households to be overconfident by deviating from the empirical frequency, this behavior is rational-
izable if the time-average of the probabilities equals its empirical frequency. Thus, the rational beliefs principle implies now the following restrictions on the beliefs

**Theorem 1.** Let the beliefs of an agent be a rational belief. Then it is restricted as follows:

- For any feasible vector parameters \((\lambda^v_g, \lambda^\omega^1_g, \lambda^\omega^2_g, \lambda^z^1_g, \lambda^z^2_g, b)\) the variance-covariance matrix \(\Omega^h\) is fully defined and not subject to choice

- \(\Omega^h\) must be a positive definite matrix. This requirement establishes a feasibility region for the vector \((\lambda^v_g, \lambda^\omega^1_g, \lambda^\omega^2_g, \lambda^z^1_g, \lambda^z^2_g, b)\). In particular, it requires \(|\lambda^v_g| \leq \sigma_v, |\lambda^\omega^i_g| \leq \sigma_\omega\) and \(|\lambda^z^i_g| \leq 1\).

- \(\{\Psi_{t+1}(g^h_t)\}\) cannot exhibit serial correlation and this restriction pins down the vector

\[
\Omega_{yy^h} = \begin{bmatrix}
\text{cov}(\tilde{\epsilon}^1_{t+1}, \tilde{\epsilon}^h_{t+1}), & \text{cov}(\tilde{\epsilon}^{1h}_{t+1}, \tilde{\epsilon}^g_{t+1}), & \text{cov}(\tilde{\epsilon}^{2h}_{t+1}, \tilde{\epsilon}^g_{t+1}), & \text{cov}(\tilde{\epsilon}^{1h}_{t+1}, \tilde{\epsilon}^{2h}_{t+1}) \\
\end{bmatrix}
\]

**Proof.** See Kurz et al. (2005b)

The beliefs are isolated by excluding \(g^h_t\) from the information in the market at \(t\). The pure belief index is defined as \(u^h_t(g^h_t)\) by using (11) and a regression as

\[ u^h_t(g^h_t) = g^h_t - r^h_t V^{-1} x_t. \]

With \(r^h_t\) defined as \(r^h_t = Cov(y, g^h),\) \(y = v^h, \omega^{1h}, \omega^{2h}, z^{1h}, z^{2h}.\)

**4 Numerical Results**

We are now providing a numerical study of the of the rational beliefs equilibrium. We use a second-order perturbation method (see e.g. Jin and Judd (2002), Schmitt-Grohe and Uribe (2004)). The numerical results indicate that under rational beliefs, the volatility of prices and consumption is higher compared to the stationary rational expectations equilibrium.
4.1 Simulation Methodology

We are describing now the simulation methodology to retrieve the stationary distribution. First, we are calculating a second-order approximation of the economy (see e.g. Jin and Judd (2002), Schmitt-Grohe and Uribe (2004)) with the program pertsov by Jin (2003). Then, we are doing a Monte Carlo simulation of the system until the averages and variances converge. The algorithm proceeds according to the following steps:

1. Find the steady-state values of the economy;

2. Compute prices, consumption, asset and money holdings given the asset holdings, money holdings of the old agents and the exogenous shocks.

3. Every 100,000 simulations: calculate the averages and standard deviations.

4. Test for convergence.

5. If we have achieved convergence: Stop, otherwise go back to step 2.

To use the perturbation method, we have to compute the steady-state values of the economy first. We denote the steady-state values with the subscript \(SS\). The steady-state price of the selling prices is \(p_{ss} = \frac{M_1}{(\bar{w}_1 + \bar{w}_2)}\). Consequently, the buying price is given by \(p_{Bss} = \frac{M_{3,t}}{(\bar{w}_1 + \bar{w}_2)}\).

In the steady-state households are indifferent between holding money and holding the asset, as both have a fixed return. As a solution to this problem, it is assumed that in the steady-state the households do not have any money holdings nor asset holdings in the steady-state.

The price for the risky asset is given by:

\[q_{ss} = \frac{\beta}{1 - \beta}.\]

The short-sale constraint \(\theta \geq \bar{\theta}\) and \(\theta_m \geq \bar{\theta}_m\) are approximated with a quadratic penalty function. This penalty function ensures that the No-Arbitrage condition is satisfied. The penalty functions are: \(\tau_S = \frac{\tau^2}{2}(\theta^h)^2\) and \(\tau_M = \frac{\tau^2}{2}(\theta^h_{m,t})^2\) with \(\tau_M = \tau_S = 0.001\).
Thus, the modified Euler equations become

\[
\frac{1}{p_t} \exp\{-\gamma^h c_t^h\} + \tau_M (\theta^h_{m,t}) = E_Q^h \left[ \exp\{-\gamma^h c_{t+1}^h\} \right] \frac{\beta}{p_{t+1}^B} |I_t|,
\]

\[
\frac{q_t}{p_t} \exp\{-\gamma^h c_t^h\} + \tau_S (\theta^h_{t}) = E_Q^h \left[ \exp\{-\gamma^h c_{t+1}^h\} \right] \beta \frac{q_{t+1} + \hat{v}_{t+1}}{p_{t+1}^B} |I_t|.
\]

For the perturbation method, we simulate now the following set of equations:

\[
v_{t+1} = (1 - \lambda_v) \bar{v} + \lambda_v v_t + \lambda^v g^h_{t+1} (\vec{u}^h_{t+1}, \epsilon) + \epsilon \bar{e}^v_{t+1},
\]

\[
\omega^1_{t+1} = (1 - \lambda_{\omega^1}) \bar{\omega}^1 + \lambda_{\omega^1} \omega^1_t + \lambda^g_{\omega^1} \hat{u}^h_{t+1} (\vec{u}^h_{t+1}, \epsilon) + \epsilon \bar{e}^\omega^1_{t+1},
\]

\[
\omega^2_{t+1} = (1 - \lambda_{\omega^2}) \bar{\omega}^2 + \lambda_{\omega^2} \omega^2_t + \lambda^g_{\omega^2} \hat{u}^h_{t+1} (\vec{u}^h_{t+1}, \epsilon) + \epsilon \bar{e}^\omega^2_{t+1},
\]

\[
z^1_{t+1} = \lambda_z z^1_t + \lambda^z_1 (v_t - \bar{v}) + \lambda^z_{11} (\omega^1_t - \bar{\omega}^1) + \lambda^z_{21} (\omega^2 - \bar{\omega}) + \lambda^z_{1h} \hat{u}^h_{t+1} (\vec{u}^h_{t+1}, \epsilon) + \epsilon \bar{e}^{z^1}_{t+1},
\]

\[
z^2_{t+1} = \lambda_z z^2_t + \lambda^z_2 (v_t - \bar{v}) + \lambda^z_{11} (\omega^1_t - \bar{\omega}^1) + \lambda^z_{22} (\omega^2_t - \bar{\omega}) + \lambda^z_{2h} \hat{u}^h_{t+1} (\vec{u}^h_{t+1}, \epsilon) + \epsilon \bar{e}^{z^2}_{t+1},
\]

where \(\epsilon\) is a scaling-factor. At the steady-state the scaling factor is \(\epsilon = 0\) and if we simulate the system we have the scaling-factor \(\epsilon = 0.0001\).

We compute the equilibria with a perturbation method. We declare a solution an equilibrium if: (i) a model is approximated by at least second order derivatives, (ii) errors in market clearing conditions and Euler equations are less than \(10^{-6}\).

For the Monte Carlo Simulation we proceed as follows. In each round we will calculate prices, consumption, asset and money holdings given the asset and money holdings of the previous round and the endowment and dividend shocks. The initial endowment of the households in asset and money holding is zero.

To test for the convergence of the system we proceed as follows: Every 100,000 rounds we compute the averages and the standard deviations and calculate the difference to the averages and standard deviations from 100,000 rounds before. We declare that the system has converged to the stationary distribution if the biggest difference is smaller than \(10^{-6}\). Thus, the algorithm can be described as follows:

### 4.2 Monetary Policy and the Volatility of Consumption

We are now studying the effects of monetary policy on the volatility of consumption. We restrict ourselves to economies with symmetric beliefs. If the beliefs are symmetric, households differ only in their state of belief.
For both households the discount factor is $\beta = 0.96$. The average endowment of the household is $\omega^h = 0.5$. And the standard deviation of the shocks is 0.005, i.e. about 1% of the average endowment. In line with the literature, we assume that the endowment shocks are very persistent and set $\lambda_{\omega}^h = 0.9$.

We study the economy under three different money supplies, $M_1 = 0.75, 1$ and $1.25$ and the monetary policy shock occurs at $t = 0$. After this initial policy shock there are no other policy shocks. Because the asset is nominal, we have to scale the dividends and the standard deviation of the dividends accordingly. Otherwise, the properties of the asset in real terms would change.

If the money supply is $M_1 = 1$, then the average dividends are 0.2 and the standard deviation of the dividend shock is 0.0105. If the money supply is $M_1 = 0.75$, then the average dividends are 0.15 and the standard deviation is 0.007875. For $M_1 = 1.25$, we have $\bar{v} = 0.25$ and $\sigma_v = 0.013125$. Furthermore, we assume that the correlation between the endowment shocks and the dividend shocks is small but negative, i.e. $\sigma_{v,\omega^h} = -0.1$. Because of the negative correlation, households want to hold the asset in order to insure themselves against income shocks.

For the parameters of the perception model, we follow Kurz et al. (2005b) and set its parameters close to the maximum possible value as implied by the definition of rationality. The chosen parameters are shown in table 1 and imply that with a positive dividend or endowment shock the households also revise their expectations upwards.

We compare the effects of a change in the money supply in an Rational Beliefs Equilibrium to an Rational Expectations Equilibrium. First, we consider the case in which households have equal risk-aversion, i.e. $\gamma = 2$. For the stationary Rational Expectations Equilibrium, we set all parameters of the perception model to 0. Thus, the households believes that the empirical distributions are the true distributions. Table 2 presents the results of the numerical simulation.

**Result 1.** *Under Rational Expectations Money is neutral.*

<table>
<thead>
<tr>
<th>$\lambda_{\omega}^v$</th>
<th>$\lambda_{\omega}^g$</th>
<th>$\lambda_{\omega}^z$</th>
<th>$b$</th>
<th>$\lambda_z$</th>
<th>$\lambda_{vz}$</th>
<th>$\lambda_{gv}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.004</td>
<td>0.001</td>
<td>0.02</td>
<td>-6</td>
<td>0.7</td>
<td>0.4</td>
<td>0.9</td>
</tr>
</tbody>
</table>
The result that a simple shift in the aggregate money supply has no real effects is not new and has been studied extensively in the literature. Under Rational Expectations the trading volume of the risky asset grows proportionally with the volatility of dividends. In addition, the trading volume on money is also increasing in the volatility of dividends. Under rational expectations, the households increase their trading volume to smooth the consumption. Thus, money is neutral.

**Result 2.** Under Rational Beliefs and homogeneous risk-aversion, money is not neutral. In particular, an increasing money supply affects the endogenous variables as follows:

\[
\begin{array}{l|c|c|c|c|c}
    p^t & q & \theta & \theta_m & c & r \\
    + & + & - & - & - & -
\end{array}
\]

A +(-) in the table means that the volatility of the variable is increasing(decreasing) with the money supply.

Furthermore, the volatility of all variables, except the volatility of the buying price, are higher under rational beliefs than under rational expectations.

The numerical results also indicate that the qualitative effects of money does not depend on the persistence of the shocks.

We are now turning to the interpretation of the results, in particular we are investigating why consumption decreases with an increase in the money supply.

The increase in the money supply has no effects on the endowment process. However, by looking at equations (8d) and (8e) one can see that the change in the volatility of the dividends affects the volatility of the market state of belief, i.e. an increase in the volatility of dividends increases the volatility of the market state of belief.

With an increasing volatility of the market state of belief the households face now a greater risk about the possible decisions of the other households. This leads to a decrease in the speculation motive. Hence, the trading volume of the households decreases. Which in turn means that the volatility of consumption decreases.

We are now turning to the case of heterogeneous risk aversion.
Result 3. If households hold rational beliefs and have different risk-aversion, than money is not neutral. In particular, an increase in the money supply affects the endogenous variables in the following way:

\[
\begin{array}{c|c|c|c|c|c|c}
 p^B & q & \theta & \theta_m & c & r \\
 + & + & - & - & - & - \\
\end{array}
\]

A + (−) in the table means that the volatility of the variable is increasing (decreasing) with the money supply.

Furthermore, under rational beliefs all endogeneous variables have a higher volatility compared to rational expectations.

If the households have different risk-aversion, under rational beliefs the volatility of prices and consumption is still higher compared to the stationary rational expectations case.

There is one significant difference to the situation in which households had equal risk-aversion. The volatility of the buying price is now higher under rational beliefs, then it is under rational expectations. However, it seems that with an increase in the money supply, the volatility of the buying price is converging to the volatility of the rational expectations case.

To sum up the discussion, as in Motoles (2001, 2003) with rational beliefs, money is not neutral. However, the propagation mechanisms are different. Here, the non-neutrality of money does not come from shocks to the money supply but a change in the volatility of financial markets. Which in turn affects the volatility of the market state of belief.

The importance of the financial markets in the propagation of monetary policy has been stressed in the literature. In the model presented here, we focussed on the effects of the beliefs on the households decisions. We found that an increasing money supply influences the beliefs of the households because of the changed volatility in the financial markets.

4.3 Beliefs and the Effects of Monetary Policy

In the previous subsection, we have seen that an expansionary monetary policy causes a decline in the volatility of consumption. But how do the beliefs of the households affect the efficiency of monetary policy?
We assume that the households have equal risk-aversion. The belief parameters of the previous section are the base case and we are varying these parameters to study the economy. The results of this simulations are depicted in tables (4)-(7).

The results of the simulations are as follows:

**Result 4.** Under rational beliefs and homogeneous risk aversion, the impact of the beliefs parameters can be summarized as follows:

\[
\begin{array}{c|c|c|c|c|c}
\lambda^g_0 & \lambda^g_0 & \lambda^g_0 & \sigma_{z_{12}} & b \\
\hline
p^B & 0 & 0 & 0 & 0 & 0 \\
q & * & 0 & + & - & - \\
\theta & * & 0 & + & - & - \\
\theta_m & * & 0 & + & - & - \\
c & * & 0 & + & - & - \\
r & * & 0 & + & - & - \\
\end{array}
\]

*Here, an \(+(-)\) indicates that an increase in the parameter increased (decreased) the volatility of the endogenous variable. A \(*\) indicates that there was no clear direction.*

The first three parameters \((\lambda^g_0, \lambda^g_0, \lambda^g_0)\) represent whether the households beliefs are optimistic or pessimistic with respect to the stationary measure. Furthermore, the parameter \(b\) represents the intensity of the fat tails in the households beliefs. Lastly, the parameter \(\sigma_{z_{12}}\) states the correlation between the households beliefs.

The first parameter we consider is \(\lambda^g_0\). With the exception of the volatility of the buying price, the volatility of the endogenous variables \((q, \theta, \theta_m, c)\) are higher if the households beliefs about the dividends is different from the stationary measure, i.e. \(\lambda^g_0 \neq 0\). In addition, the volatility is higher if the households are optimistic rather than pessimistic. If the households are optimistic with respect to the dividends, they are investing less in the risky asset. Thus, they get punished by not saving enough and to finance consumption they have to borrow money. Because of the wrong savings decisions, their consumption volatility is much higher.

However, his beliefs with respect to the endowment are not important in determining the trading volume and the volatility of consumption. Thus, the beliefs
about the future endowments do not play an important role in the decisions of the households.

We are now turning to the parameter $\lambda_g$, the parameter that determines whether the households are optimistic or pessimistic about the market state of belief. Or, to put it in other words, the beliefs about the decisions of the other households.

Next, we are discussing the correlation of the beliefs. A higher correlation of beliefs induces a lower volatility of the endogenous variables, because with a higher correlation of beliefs the motive for speculation on the beliefs on the other households decreases. The case of $\sigma_{z1z2} = 1$ would represent an economy in which households hold the same beliefs and thus we would have the same volatility as in the case of rational expectations.

Lastly, we consider the parameter $b$. This parameter determines the intensity of the fat tails in the households’ beliefs. In other words, a smaller $b$ means that we have a higher degree of overconfidence as the households put more probability weight on extreme events. If the households put more weight on extreme events, they demand more insurance. However, as they demand too much insurance, the trading volume increases.

In conclusion, we have seen that not all parameters affect the efficiency of monetary policy in decreasing the volatility of consumption. While the expectation about the future endowment did not affect the distribution, all other variables did affect the allocation in equilibrium.

5 Conclusion

This paper studied the impacts of a simple monetary policy on the standard deviation of consumption. Households were able to invest into a risky asset which paid nominal dividends and hold money as a store of value. The transactions role of money was introduced with a cash-in-advance constraint. Furthermore, the households held rational beliefs.

We found that an increase in the money supply decreased the volatility of consumption. Furthermore, the standard deviation of the households money holdings and the transaction volume of the risky asset also decreased. As long as the house-
holds had equal risk-aversion, the beliefs had only a small impact on the prices in the economy.

The intuition behind the result that an increased money supply decreases the volatility of consumption is as follows. With an increase in the money supply the households faced a greater risk with respect to the other households’ action. Which in turn led to a decline in the trading volume and thus in the volatility of consumption.

Thus, the result gives us some new insight about the transmission of monetary policy. Here, monetary policy affects the households through the financial market. As opposed to the traditional view, the transmission mechanism is not a wealth effect (see e.g. Lettau et al. (2002)), but an beliefs effect.

This view of the monetary transmission is confirmed that in the presented model the parameters for the households perception of the dividends had an effect but not the ones for the endowment of the household.

Although the results seems to imply a straightforward recommendation for monetary policy, one has to be careful. The result depends on the fact that the households have heterogeneous beliefs. Unlike prices, the beliefs of the households are not observable for the policy makers.

We did not study the question if monetary policy improves the welfare in the economy. As it has been pointed out by Nielsen (2009a,b) under heterogeneous beliefs, a policy that can be pareto-optimal ex-ante must not be optimal ex-post.

References


Sanford Grossmann and Laurence Weiss. A transactions-based model of the mon-


A Tables

<table>
<thead>
<tr>
<th>$M_1$</th>
<th>0.75</th>
<th>1</th>
<th>1.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^B$</td>
<td>0.009</td>
<td>0.012</td>
<td>0.014</td>
</tr>
<tr>
<td>$q$</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.009</td>
<td>0.010</td>
<td>0.009</td>
</tr>
<tr>
<td>$\theta_m$</td>
<td>0.006</td>
<td>0.010</td>
<td>0.09</td>
</tr>
<tr>
<td>$c$</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
</tr>
</tbody>
</table>

Table 2: The effects of an increase in the money supply on the trading volume and volatility of prices and consumption if households hold rational expectations.

<table>
<thead>
<tr>
<th>$M_1$</th>
<th>0.75</th>
<th>1</th>
<th>1.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^B$</td>
<td>0.009</td>
<td>0.012</td>
<td>0.014</td>
</tr>
<tr>
<td>$q$</td>
<td>0.021</td>
<td>0.021</td>
<td>0.021</td>
</tr>
<tr>
<td>$\theta$</td>
<td>1.734</td>
<td>1.096</td>
<td>0.688</td>
</tr>
<tr>
<td>$\theta_m$</td>
<td>0.354</td>
<td>0.203</td>
<td>0.130</td>
</tr>
<tr>
<td>$c$</td>
<td>0.043</td>
<td>0.036</td>
<td>0.029</td>
</tr>
</tbody>
</table>

Table 3: The effects of an increase in the money supply on the trading volume and volatility of prices and consumption if households have rational beliefs.

<table>
<thead>
<tr>
<th>$\lambda_v^g$</th>
<th>$-0.004$</th>
<th>0</th>
<th>0.004</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>0.75</td>
<td>1</td>
<td>1.25</td>
</tr>
<tr>
<td>$p^B$</td>
<td>0.009</td>
<td>0.012</td>
<td>0.015</td>
</tr>
<tr>
<td>$q$</td>
<td>0.020</td>
<td>0.020</td>
<td>0.020</td>
</tr>
<tr>
<td>$\theta_m$</td>
<td>1.658</td>
<td>1.046</td>
<td>0.659</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.337</td>
<td>0.193</td>
<td>0.1241</td>
</tr>
<tr>
<td>$c$</td>
<td>0.042</td>
<td>0.037</td>
<td>0.027</td>
</tr>
</tbody>
</table>

Table 4: Trading volume, consumption and volatility of prices with $\lambda_v^g = -0.004, 0, 0.004$. 

25
### Table 5: Trading volume and volatility of prices with $\lambda_g = -0.001, 0, 0.001$

<table>
<thead>
<tr>
<th>$\lambda_g$</th>
<th>-0.001</th>
<th>0</th>
<th>0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>0.75 1 1.25</td>
<td>0.75 1 1.25</td>
<td>0.75 1 1.25</td>
</tr>
<tr>
<td>$P^B$</td>
<td>0.088 0.012 0.0146</td>
<td>0.0875 0.012 0.015</td>
<td>0.009 0.012 0.014</td>
</tr>
<tr>
<td>$q$</td>
<td>0.021 0.022 0.022</td>
<td>0.021 0.021 0.021</td>
<td>0.021 0.021 0.021</td>
</tr>
<tr>
<td>$\theta_m$</td>
<td>1.780 1.126 0.706</td>
<td>1.756 1.111 0.696</td>
<td>1.734 0.203 0.688</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.363 0.209 0.134</td>
<td>0.358 0.206 0.132</td>
<td>0.354 1.096 0.130</td>
</tr>
<tr>
<td>$c$</td>
<td>0.044 0.037 0.030</td>
<td>0.044 0.037 0.029</td>
<td>0.043 0.036 0.029</td>
</tr>
</tbody>
</table>

### Table 6: Trading volume and volatility of prices with $\lambda_g = -0.02, 0, 0.02$

<table>
<thead>
<tr>
<th>$\lambda_g$</th>
<th>-0.001</th>
<th>0</th>
<th>0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>0.75 1 1.25</td>
<td>0.75 1 1.25</td>
<td>0.75 1 1.25</td>
</tr>
<tr>
<td>$P^B$</td>
<td>0.009 0.012 0.015</td>
<td>0.088 0.012 0.015</td>
<td>0.009 0.012 0.014</td>
</tr>
<tr>
<td>$q$</td>
<td>0.019 0.022 0.020</td>
<td>0.019 0.019 0.020</td>
<td>0.021 0.021 0.021</td>
</tr>
<tr>
<td>$\theta_m$</td>
<td>2.751 1.547 1.082</td>
<td>3.587 2.253 1.367</td>
<td>1.734 1.096 0.688</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.562 0.287 0.205</td>
<td>0.722 0.414 0.260</td>
<td>0.354 0.203 0.130</td>
</tr>
<tr>
<td>$c$</td>
<td>0.068 0.051 0.044</td>
<td>0.089 0.074 0.056</td>
<td>0.043 0.036 0.029</td>
</tr>
</tbody>
</table>

### Table 7: Trading volume and volatility of prices with $\sigma_{z_1 z_2} = 0.9, 0.75, 0.6$

<table>
<thead>
<tr>
<th>$M_1$</th>
<th>0.75 1 1.25</th>
<th>0.75 1 1.25</th>
<th>0.75 1 1.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^B$</td>
<td>0.009 0.012 0.014</td>
<td>0.009 0.012 0.015</td>
<td>0.009 0.012 0.015</td>
</tr>
<tr>
<td>$q$</td>
<td>0.021 0.021 0.021</td>
<td>0.018 0.018 0.018</td>
<td>0.019 0.019 0.019</td>
</tr>
<tr>
<td>$\theta_m$</td>
<td>1.734 1.096 0.688</td>
<td>1.449 0.917 0.576</td>
<td>1.580 0.999 0.627</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.354 0.203 0.130</td>
<td>0.296 0.170 0.109</td>
<td>0.322 0.185 0.119</td>
</tr>
<tr>
<td>$c$</td>
<td>0.043 0.036 0.029</td>
<td>0.036 0.031 0.024</td>
<td>0.039 0.033 0.026</td>
</tr>
</tbody>
</table>

### Table 8: Trading volume and volatility of prices with $b = -6, -4, 0$

<table>
<thead>
<tr>
<th>$M_1$</th>
<th>0.75 1 1.25</th>
<th>0.75 1 1.25</th>
<th>0.75 1 1.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^B$</td>
<td>0.009 0.012 0.014</td>
<td>0.009 0.012 0.0146</td>
<td>0.009 0.012 0.015</td>
</tr>
<tr>
<td>$q$</td>
<td>0.021 0.021 0.021</td>
<td>0.013 0.014 0.014</td>
<td>0.001 0.001 0.0013</td>
</tr>
<tr>
<td>$\theta_m$</td>
<td>1.734 1.096 0.688</td>
<td>1.115 0.706 0.4451</td>
<td>0.085 0.089 0.0867</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.354 0.203 0.130</td>
<td>0.227 0.131 0.084</td>
<td>0.007 0.009 0.011</td>
</tr>
<tr>
<td>$c$</td>
<td>0.043 0.036 0.029</td>
<td>0.028 0.024 0.019</td>
<td>0.006 0.007 0.007</td>
</tr>
</tbody>
</table>