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Abstract The well-known Blackwell's theorem states the equivalence of statistical informativeness and economic valuableness. Çelen (2012) generalizes this theorem, which is well-known for subjective expected utility (SEU), to maxmin expected utility (MEU) preferences. We demonstrate that the underlying definition of the value of information used in Çelen (2012) is in contradiction with the principle of recursively defined utility. As a consequence, Çelen's framework features dynamic inconsistency. Our main contribution consists in the definition of a value of information for MEU preferences that is compatible with recursive utility and thus respects dynamic consistency.

Keywords Value of information; Maxmin expected utility; Recursive utility

JEL D81; D83; D84

1 Introduction

For decades economists have been studying the relationship between decision-making under uncertainty and the so-called value of information. A famous and well-known result in this context is Blackwell's theorem (Blackwell 1953) stating that an experiment is more valuable than another if and only if the same experiment is more informative than the latter. In order to obtain this

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equivalence (e.g. Crémer 1982), a standard assumption has been that decision-makers are subjective expected utility (SEU) maximizers, cf. Savage (1954).

The objective of Çelen (2012) is to extend the Blackwell theorem to MEU preferences, which were axiomatized by Gilboa and Schmeidler (1989). In this note we demonstrate that Çelen's proof relies on a value of information for MEU preferences that is not defined via backward induction and thus is incompatible with the intertemporal extension of MEU by Epstein and Schneider (2003). In particular, optimal strategies in Çelen's framework prescribe decisions conditional on signal realizations that an MEU decision-maker will not find optimal to adhere to once those signal realizations have been observed. In this sense, Çelen's framework features dynamic inconsistency. Our contribution is to define a value of information that is compatible with the recursive intertemporal formulation of MEU preferences by Epstein and Schneider (2003).

2 Framework and definition of the value of information in Çelen (2012)

In the following, we adopt Çelen's framework and notation. Let $\Omega := \{\omega_1, \ldots, \omega_n\}$ be the finite set of *states* and $X := \{a_1, \ldots, a_\chi\}$ the finite set of *actions* available to a decision-maker. Moreover, let $\Delta(\Omega)$ and $\Delta(X)$ be the set of all probability distributions defined on Ω and X, respectively. Let further $u : \Omega \times X \to \mathbb{R}$ be a *utility function* and $\boldsymbol{u}$ with $u_{ij} = u(\omega_i, a_j)$ the corresponding utility matrix. An SEU decision-maker is characterized by $(\boldsymbol{\pi}, \boldsymbol{u})$, where $\boldsymbol{\pi} \in \Delta(\Omega)$ is a prior over the states.

An *experiment* is a tuple $(S, \boldsymbol{p})$ with the signal space $S = \{s_1, \ldots, s_\sigma\}$ and the Markov matrix $\boldsymbol{p}$ with $p_{ij} = \Pr(s_j|\omega_i)$ for $s_j \in S$. Çelen introduces a *strategy* as a vector valued mapping $f : S \to \Delta(X)$, thus characterizing all (mixed) actions the decision maker plans to take after observing certain signal realizations s. The $\sigma \times \chi$-matrix $\boldsymbol{f}$ is defined such that $(f_{i1}, \cdots, f_{i\chi}) := f(s_i)$.

In this framework, Çelen determines the *value of the experiment* $(S, \boldsymbol{p})$ *for a given strategy* $\boldsymbol{f}$ as

$$\mathcal{U}^{\boldsymbol{f}}_{(\pi,u)}(S, \boldsymbol{p}) = \sum_j \Pr(s_j) \sum_i \Pr(\omega_i|s_j) \sum_k f_{jk} u(\omega_i, a_k) \tag{1}$$

$$= \sum_j \sum_i p_{ij} \pi_i \sum_k f_{jk} u_{ik} \qquad \text{(by Bayes' rule)}\ . \tag{2}$$

With a strategy $\boldsymbol{f}^*$ maximizing (2), Çelen defines $\mathcal{U}^*_{(\pi,u)}(S, \boldsymbol{p}) = \mathcal{U}^{\boldsymbol{f}^*}_{(\pi,u)}(S, \boldsymbol{p})$ as the value of the experiment for an SEU decision-maker.

Building on this, Çelen extends the definition of the value of an experiment to the class of MEU preferences. For that purpose, he characterizes an MEU decision-maker by (A, u), where $A \subset \Delta(\Omega)$ is a convex and compact *set* of priors. As a counterpart of $\mathcal{U}^*_{(\pi,u)}(S, \boldsymbol{p})$, he defines

$$\mathcal{W}^*_{(A,u)} = \max_{\boldsymbol{f}} \min_{\boldsymbol{\pi} \in A} \mathcal{U}^{\boldsymbol{f}}_{(\pi,u)}(S, \boldsymbol{p}) \tag{3}$$

as the value of an experiment $(S, \boldsymbol{p})$ for an MEU decision-maker. It is expression (3) that Çelen relies on in his proof of the generalized Blackwell's theorem.

3 A recursively defined MEU value of information

It is insightful to note that Çelen's framework essentially constitutes an intertemporal setting with two periods. In the second period, *after* observing a signal realization, the decision maker takes a (mixed) action. In the first period, *before* observing the signal realization, the value of the experiment $(S, \boldsymbol{p})$ is determined. Çelen accounts for the intertemporal structure insofar as he considers strategies, that is complete contingent plans for appropriate play after observing signal realizations.

His formulation, however, is in contrast to the usual intertemporal formulation of MEU preferences that is provided by Epstein and Schneider (2003). One of the main characteristics of their recursive definition of intertemporal MEU is the compatibility with backward induction. We follow the approach in Epstein and Schneider (2003) and develop an alternative definition of the value of information for MEU preferences.

According to backward induction, the first step to define a value of information is to determine the value of the final decision given that the decision maker chooses an optimal action for all possible signal realizations s_j, $j = 1, \ldots, \sigma$. For MEU preferences this value is

$$V(\mathcal{M}_j) = \max_{g \in \Delta(X)} \min_{\boldsymbol{\mu} \in \mathcal{M}_j} \mathbb{E}_{\boldsymbol{\mu}}[u](g) , \tag{4}$$

where $\mathbb{E}_{\boldsymbol{\mu}}[u](g) = \sum_{k,i} g_k \mu_i \, u_{ik}$ denotes the expected utility under action $g \in \Delta(X)$ and ex-post belief μ. The optimal action is determined considering the worst posterior $\mu \in \mathcal{M}_j$. Formally, the set of posteriors is $\mathcal{M}_j = \{\boldsymbol{\pi}(\cdot|s_j) \, : \, \boldsymbol{\pi} \in A\}$, where $\boldsymbol{\pi}(\cdot|s_j)$ denotes the conditional probability of the prior $\boldsymbol{\pi} \in \Delta(\Omega)$ given the signal s_j. We obtain $\boldsymbol{\pi}(\cdot|s_j)$ via Bayes' rule and update each prior $\boldsymbol{\pi}$ in this way.[1]

[1] Epstein and Schneider (2007) show that further restrictions on the set $\mathcal{M}$ can be made. For the sake of simplicity, you may think of full Bayesian updating.

Building on (4), we can define the value of the experiment $(S, \boldsymbol{p})$ for MEU preferences as

$$\mathcal{V}_{(A,u)} = \min_{\boldsymbol{\pi} \in A} \sum_{i,j} \pi_i p_{ij} V(\mathcal{M}_j) \ . \tag{5}$$

As usual, the value of information is obtained by taking the expectation over all possible signal realizations. Due to MEU preferences the value of the experiment is the worst of those expectations. This alternative way of defining the value of an experiment is in line with the intertemporal model of recursive utility under multiple priors as introduced in Epstein and Schneider (2003, 2007).

The key characteristic of (5) is that optimal actions are determined with the maxmin rule for each signal realization s_j individually. In particular, the worst posterior in (4) in general depends on the signal realization s_j. This is in contrast to (3). By following the derivation of the SEU counterpart, essentially the step from (1) to (2), Çelen silently assumes that the worst prior from the ex-ante perspective coincides with the preimage of *all* worst posteriors, irrespective of the signal realization. For the SEU decision-maker this argumentation is innocent as there is a unique prior, and thus a unique posterior as well. For the MEU decision-maker, however, this argument is in conflict with backward induction.

In the appendix, we demonstrate that the conflict of Çelen's framework with intertemporal recursive utility can be made even more concrete. We provide an example in which the optimal strategy derived in Çelen's framework prescribes actions that are different from what an MEU decision-maker will actually do after observing those signals realizations.[2] This supports our claim that the value of information for MEU preferences should be defined by (5). By construction, our definition of the value of information is compatible with dynamic consistency.

[2] One could think that the reason we observe this form of dynamic inconsistency is the missing assumption of *rectangularity* of the prior set, a key assumption in Epstein and Schneider (2003) to ensure dynamic consistency within an intertemporal setting of recursive utility. But this is not the case. Even though Çelen's setting is not fully transferable to the setting of Epstein and Schneider, in particular the analysis in Epstein and Schneider (2007) suggests that rectangularity is no issue in this setting, simply because the learning process is defined via conditional one-step-ahead conditionals, as required by Epstein and Schneider (2003). The reason for the violation of dynamic consistency in Çelen's framework is that intertemporal utility is defined in a non-recursive way, thus incompatible with dynamic consistency right from the start.

4 Results and Discussion

We have shown that Çelen's proof of Blackwell's theorem only applies to a value of information that is defined in a non-recursive utility framework. We have offered a definition for the value of information derived via backward induction, thus compatible with the dynamic consistent intertemporal axiomatization of Epstein and Schneider (2003). Consequently, we argue that the proof of Blackwell's theorem should deal with expression (5) as the definition of the value of information for MEU preferences. This proof is still pending.

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A Example demonstrating the dynamic inconsistency of Çelen's framework

We restrict the number of states of the world $\Omega = \{\omega_1, \omega_2\}$, actions $X = \{a_1, a_2\}$ and signal realizations $S = \{s_1, s_2\}$ to two. Payoffs are specified by $u_{11} = 1$, $u_{12} = -1$, $u_{22} = 2$ and $u_{21} = 0$. This is a simple example of a setting in which the decision maker wants to learn the true ω because action a_1 is optimal if $\omega = \omega_1$ and action a_2 is optimal if $\omega = \omega_2$.

For the signal likelihood $\lambda = p_{11} = p_{22}$ we assume $1/2 < \lambda < 3/4$ and specify the set of priors as $A = \{(\pi_1, 1 - \pi_1) \,:\, 1/4 \leq \pi_1 \leq 3/4\}$.

Using equation (2), we obtain that the optimal strategy $\boldsymbol{f}^*$ in Çelen's framework is

$$f^*(s_1) = (1, 0) \quad , \quad f^*(s_2) = (1/2, 1/2) \ . \tag{6}$$

In words, the optimal strategy in the Çelen framework consists of taking action a_1 if $s = s_1$ and mixing over actions a_1 and a_2 with equal weights if $s = s_2$.

Next, we demonstrate that an MEU decision-maker operating with the principle of backward induction would deviate from $\boldsymbol{f}^*$ as soon as the signal materializes. The decision rule after observing a signal realization s_j is given in (4), where g is a randomization over actions a_1 and a_2, and $\mathcal{M}_j \subset \Delta(\Omega)$ is the set of posteriors that depends on the set of priors A, the likelihood p, and the signal realization s_j observed. After simple algebra we get

$$g^*(s_1) = (3/4, 1/4) \quad \text{and} \quad g^*(s_2) = (3/4, 1/4) \ . \tag{7}$$

In words, the optimal action of the MEU decision-maker, both after receiving $s = s_1$ and $s = s_2$, is to mix over actions a_1 and a_2 with the ratio 3 to 1. This is in contrast to the behavior prescribed in (6). Our example thus demonstrates the dynamic inconsistency in Çelen's framework.