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Abstract

The literature on environmental R&D frequently studies innovation as a two-stage process, with a single R&D event leading from a conventional polluting technology to a perfectly clean backstop. We allow for uncertainty in innovation in that the new technology may turn out to generate a new pollution problem. R&D may therefore be optimally undertaken more than once. Using and externding recent results from multi-stage optimal control theory, we provide a full characterization of the optimal pollution and R&D policies. The optimal R&D program is strictly sequential and has an endogenous stopping point. Uncertainty drives total R&D effort and its timing.

JEL classification: Q55, Q53, O32, C61
Keywords: stock pollution, backstop technology, multi-stage optimal control, pollution thresholds, uncertainty

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1 Introduction

Backstop technologies are a common point of reference in dynamic models of the environment and natural resources, beginning with the influential study by Nordhaus (1973) on exhaustible sources of energy. There, a backstop technology essentially involves access to a resource with an infinite stock. More recently, in the context of the expanding literature on the economics of stock pollutants, "*the backstop*" has become a shorthand for perfectly clean technologies that do not suffer from a stock pollution problem. In both cases, the backstop allows the decision maker to escape a binding constraint forever.

The existing literature on backstops offers optimal timing rules regarding the phasing in of a backstop in a variety of different settings and under varying degrees of uncertainty. In the area of non-renewable resources, Dasgupta and Heal (1974) study optimal exhaustion when the arrival time of the exogenously provided backstop technology is stochastic. Hung and Quyen (1993) endogenize the decision when to invest in R&D in a setting where the length of time required to develop the backstop is uncertain. Tsur and Zemel (2003) develop a deterministic model with the difference that the backstop can be continuously improved through additional R&D. Just et al. (2005) provide a stochastic, but discrete analysis of a similar problem. In the context of stock pollution Baudry (2000) applies real options theory in a setting where the backstop arrives stochastically after R&D is commenced; and Fischer et al. (2004) consider the optimal investment path for an existing clean backstop technology.

One type of uncertainty that has not been considered so far in the literature is uncertainty about the characteristics of new technologies. Commonly, models rely on an assumption of technological certainty in R&D: If the backstop is not already available, the next technology to be invented will always constitute a backstop. A well defined R&D investment will therefore always generate a final resolution of the intertemporal constraint. Looking at the empirical record, this idea is at least arguable. Two prominent examples illustrate this point: In the case of ozone-depleting chlorofluorocarbons (CFCs), newly developed substitutes such as HCFC-123 were demonstrated to feature a more benign stratospheric chemistry, but also shown to imply a different stock pollution problem on account of decaying into toxic pollutants such as trifluoroacetic acid. The primary substitute for fossil fuels, nuclear energy, may provide advantageous properties with respect to exhaustibility, but involves the production of long-lived stocks of radioactive waste. These are only two illustrations of a more general observation, namely that technologies developed in response to binding intertemporal constraints may relax those constraints, but will not always allow decision-makers to escape them indefinitely. In such a situation, investments in R&D have to be considered under the premise that the arrival of a backstop is only one of two possible outcomes of the innovation process. Instead, R&D may generate a technology that is novel, but has stings attached in the form of an intertemporal pollution dynamic of its own. The possibility of the intertemporal constraint recurring even after R&D resources have been expended is the possibility of technological '*boomerangs*' that the title refers to.

In this paper, we study the implications of allowing for technological uncertainty over innovation outcomes on optimal R&D timing, choosing the context of stock pollutants as a setting. To model technological uncertainty, we consider a decision-maker who attaches a probability to the possibility that new technologies may not turn out to be the clean backstop that will solve the pollution problem once and for all, and we allow these beliefs about the probabilities to become decision-relevant. This small change in the assumptions about the decision-maker's view about the likely environmental characteristics of new technologies has important repercussions for his thinking about pollution policies and R&D timing. The change extends the set of possible future states of the world to situations where new technologies turn out to have undesirable properties. This means that R&D may have to be undertaken more than once in order to solve the pollution problem. In fact, the possibility of lengthy sequences of failures to find a backstop despite R&D investment can no longer be excluded by the planner. This has repercussions for the optimal pollution policy since future costs of current emissions depend on the degree of uncertainty over the discovery of a backstop.

While it seems clear that the possibility of receiving (possibly multiple) technologies of the '*boomerang*' type in the quest for a backstop should change the optimal prescriptions both for environmental and for technology policy, the precise nature of these changes is less obvious. Should the policymaker's respond to the presence of technological uncertainty with higher or lower R&D efforts? Should R&D be carried out on a large scale right at the start (frontloaded) or spread out over time? How should the policymaker respond to the invention of a 'boomerang' technology - with more R&D right away or with waiting? Should R&D ever stop even though a backstop has not been found yet? We develop a specific setting in which these questions can be answered on the basis of analytical solutions. This is in order to develop a first intuition on the impact of technological uncertainty on optimal R&D and in order to provide a building block for considering more general cases in the future.

The simple and tractable model consists of a production sector producing a single product up to a fixed output constraint, with one technology of the boomerang type available *ab initio*. Production generates a profile of technology-specific pollutants. Once a backstop is available, that part of production carried out using the backstop will produce no pollution at all. Damages are convex in the stock of each pollutant and additive across pollutants, giving rise to gains from diversification in pollutants and hence incentives for conducting R&D even when a backstop is not feasible. To retain a clear focus on the role of uncertainty, other important R&D drivers whose impacts have been established in the literature are excluded from the analysis. R&D has a deterministic component in that at any given time, a new technology with zero stock of initial pollution can be provided at a fixed cost. What is uncertain, however, are the environmental characteristics of the new technology. Under the decision-maker's beliefs, R&D carried out at a given point in time will fail to generate a backstop with a certain probability and will generate a technology involving a new stock pollutant instead. Given this setting, we study the optimal timing of R&D and the optimal pollution policy.

In order derive the optimal R&D trajectory we utilize recent results on multi-stage optimal control with infinite horizons. This technique allows us to capture a process of technological evolution in which new technologies are added in a discrete fashion while allowing for more than one technology to be added at any given point in time. In addition to applying this technique to the question of optimal R&D trajectories, we present - to our knowledge - the first application of this technique to a situation characterized by uncertainty over the properties of the next stage of the optimal control problem. This involves a suitable modification of the necessary conditions derived by Makris (2001) and Tomiyama (1985).[1]

Our key findings are that in our setting the optimal R&D program is (i) strictly sequential in the sense that at most one technology is developed at any given point in time and (ii) has an endogenous stopping point. There is a constant pollution stock threshold level that triggers research and is above the long run steady state of pollution stocks (overshooting). Technological uncertainty affects both the optimal timing and the maximum size of the technology portfolio. The optimal pollution policy becomes more sophisticated if research fails to deliver a backstop technology.

The structure of the paper is as follows: In the next section, we describe the model set-up. Section 3 develops the optimal pollution policy for a given number of technologies. In section 4 we study the optimal timing of R&D under

[1]For a more formal treatment of deterministic infinite horizon multi-stage optimal control problems see Babad (1995).

technological uncertainty and we conclude in section 5.

2 The Model

The model consists of two fundamental components, one describing the nature of the stock pollution problem and the other the process of innovation. Jointly, they describe the social planner's problem of developing a simultaneous environmental and innovation policy under technological uncertainty.

The environmental side of our model consists of standard pollution stock dynamics common in this literature (for example Fischer et al. (2004), Baudry (2000)). At time t, there are $n\left(t\right)$ different potential pollutants $i \in \{1, ..., n\left(t\right)\}$ with associated stock levels $S_i\left(t\right)$ with stock dynamics of the type:

$$\dot{S}_i(t) = \alpha_i q_i(t) - \delta_i S_i(t) \tag{1}$$

with α_i denoting the rate of accumulation on the basis of emissions of volume q_i and δ_i denoting the rate of decay in the stock of pollutant i.

Pollutants are technology-specific and, in the interest of tractability, do not interact with each other. Hence, i denotes both the technology and the single pollutant generated by this technology. The pollution damage function is additively separable in the square of individual stocks of pollutants such that pollution damage $D\left(S_1(t), ..., S_{n(t)}(t)\right)$ caused by stocks $S_1(t)$ to $S_{n(t)}(t)$ at time t is

$$D(S_1(t), ..., S_{n(t)}(t)) = \sum_{i=1}^{n(t)} \frac{d_i}{2} S_i(t)^2 \tag{2}$$

with d_i denoting the marginal damage coefficient of pollutant i.

In order to retain a clear focus on an analytical assessment of the impact of technological uncertainty, the model contains some important simplifications regarding heterogeneity of pollutants and the shape of the social welfare function: With the exception of the backstop, technologies and pollutants respectively are assumed to be symmetric in terms of rate of accumulation $\alpha_i = \alpha$, rate of decay $\delta_i = \delta$, and the marginal damage coefficient $d_i = d$. The backstop on the other hand, representing a 'perfectly clean' technology, is characterized by zero damages and no accumulation such that $d_B = 0$ and $\alpha_B = 0$. For all technologies, costs are assumed symmetric and zero such that $c_i\left(q_i, t\right) = 0$. Technologies are perfect substitutes and symmetric in terms of net marginal benefits which are normalized to 1 per unit of output. Aggregate output is exogenously bounded from above as in Baudry (2000). This is an indirect way of taking capital stock constraints into account.

$$\sum_{i=1}^{n(t)} q_i(t) \leq 1 \tag{3}$$

$$0 \leq q_i(t) \leq 1 \quad , \forall i \in \{1, ..., n\left(t\right)\} \tag{4}$$

The symmetry of the technologies in terms of the production-pollution side of the model then provides a simplified instantaneous welfare function of the form

$$W(t) = \sum_{i=1}^{n(t)} \left[q_i(t) - \frac{d_i}{2} S_i(t)^2 \right] \tag{5}$$

in which non-backstop technologies now differ in terms of vintage only and the backstop technology differs in terms of damage intensity.

Innovation is modeled as follows: At any time t, society can choose to spend resources $R\left(t\right)$ which will make available instantaneously and with certainty the

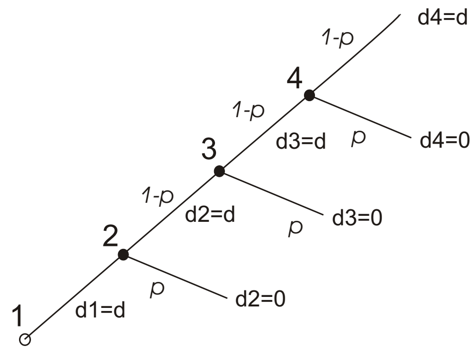

Figure 1: Potential Sequence of Innovations

$n+1^{st}$ technology. The point in time when the $n+1^{st}$ technology is developed is denoted by t_{n+1}. The number of technologies $n(t)$ available for production at t therefore depends on the sequence of past investments $\{t_1, ..., t_n\}$. The environmental characteristics of the new technology are not known prior to its arrival. With probability p, the $n+1^{st}$ technology turns out to constitute a technology of the backstop type. In the event, the number of technologies remains fixed from then on as there is no further rationale for resources to be spent on R&D in a setting where technologies are otherwise perfect substitutes. With probability $(1-p)$, the $n+1^{st}$ technology is of the boomerang type and involves the generation of a novel, technology-specific pollutant (see Figure 1). In this case, knowing that a *'boomerang'* has been produced, the social planner might decide to develop a further technology right away. Hence, in principle it is possible that more than one technology is developed at any given point in time.

All new technologies start with an initial stock of pollution $S_n(t_n) = 0$ and can at once be used at any level of intensity.[2] For convenience, we assume that the current cost of R&D is independent of time such that $R(t) = R$ and that initially, one technology is available such that $n(0) = 1$. Furthermore, we assume that there is an arbitrarily large but finite number of potential technological solutions M that can possibly be developed. Each of these solutions is a simple lottery. At the instant they are converted into technologies by R&D they materialize either as a backstop (with probability p) or as a *'boomerang'* (with probability $1 - p$). Hence, p is independent of both the maximum number of technologies feasible, M, and of the number of technologies already developed, n.

The social planner's problem is to maximize the expected value of net welfare from production over an infinite time horizon, subject to the effects of stock pollution and subject to an R&D process that can produce backstops or '*boomerangs*'. Its choice variables are on the one hand the production intensities $q_i(t)$ of the currently available technologies $i \in \{1, ..., n(t)\}$ and on the other hand the timing of R&D activities $\{t_2, t_3...\}$ that expand the set of available technologies $n(t)$ from $n(0) = 1$ up to a finite $N \leq M$ that is also endogenously

[2] We therefore do not study questions about the optimal accumulation of technology specific capital (as e.g. Fischer et al. (2004)).

determined. The problem is then

$$
\begin{aligned}
\max_{\{q_i(t)\},\{t_2,t_3,\dots,t_N\},\{N\}} = & \int_0^{t_2} e^{-rt}\left[\left(q_1 - \frac{d_1}{2}S_1^2\right)\right]dt - e^{-rt_2}R \\
& + p\int_{t_2}^{\infty} e^{-rt}\left[\sum_{i=1}^{2}\left(q_i - \frac{d_i}{2}S_i^2\right)\right]dt \\
& + (1-p)\left\{\int_{t_2}^{t_3} e^{-rt}\left[\sum_{i=1}^{2}\left(q_i - \frac{d}{2}S_i^2\right)\right]dt - e^{-rt_3}R\right. \\
& + p\int_{t3}^{\infty} e^{-rt}\left[\sum_{i=1}^{3}\left(q_i - \frac{d_i}{2}S_i^2\right)\right]dt \\
& + (1-p)\left\{\int_{t_2}^{t_3} e^{-rt}\left[\sum_{i=1}^{3}\left(q_i - \frac{d}{2}S_i^2\right)\right]dt\right. \\
& + p\ldots \\
& \left.\left. + (1-p)\ \int_{t_N}^{\infty} e^{-rt}\left[\sum_{i=1}^{N}\left(q_i - \frac{d}{2}S_i^2\right)\right]dt\right\}\right\} \qquad (6)
\end{aligned}
$$

subject to conditions (1), (3), (4) and the transversality condition

$$\lim_{t\to\infty} H_N^*(t) = 0. \qquad (7)$$

To sum up, the nature of the planner's problem describes a situation in which the choice of pollution policy and R&D policy are linked in two ways. Firstly, the past history of R&D determines the planner's current degrees of freedom in allocating production shares to different technologies. Secondly, depending on research success regarding the backstop, additional R&D may optimally be undertaken or not.

The solution to the social planner's problem involves characterizing the control processes of production shares and R&D timing given the state processes of stock dynamics. The heuristic strategy involves separating the problems into an optimal pollution policy given the number and type of technologies already developed and the optimal R&D policy that determines the extension of the set of technologies at any given point in time.

3 The Optimal Pollution Policy

With uncertainty only entering at instants of innovation, the optimal pollution policy *between* any two innovation events is a standard deterministic Markov-process where the number of state variables equals the number of available technologies. Conditional on the number and type of technologies and the pollution stocks at the beginning of the considered planning period, the optimal policy can be derived. This is done in this section while the optimal R&D policy is studied in Section 4. Note that while studying the optimal pollution policy the number of technologies remains fixed at $n = n(t_i)$ for all $t \in [t_i, t_{i+1})$, $i = \{1, ..., N\}$, where $t_1 = 0$ is the arrival time of the first (free) technology.

Given the number of technologies n and their pollution stock levels $S_i\left(t\right)$, the Hamiltonian of this problem is

$$H = e^{-rt}W(t) + \sum_{i=1}^{n} \left[\mu_i(t)\left(\alpha q_i(t) - \delta S_i(t)\right)\right] + e^{-rt}\kappa_n(t)\left(1 - \sum_{i=1}^{n} q_i(t)\right)$$

where μ_i is the shadow price of pollution stock S_i and κ_n is the shadow price of the output constraint (3). The corresponding first order conditions are

$$\frac{\partial H}{\partial q_i} = e^{-rt} + \alpha\mu_i(t) - e^{-rt}\kappa_n(t) = 0 \tag{8}$$

$$-\frac{\partial H}{\partial S_i} = e^{-rt}dS_i(t) + \delta\mu_i(t) = \dot{\mu}_i. \tag{9}$$

Condition (8) gives rise to the following switching function

$$\sigma_i(t) = e^{-rt} + \alpha\mu_i(t) - e^{-rt}\kappa_n(t) \begin{cases} < 0 & , \quad q_i(t) = 0 \\ = 0 & , \quad q_i(t) = q_i^*(t) \\ > 0 & , \quad q_i(t) = 1 \end{cases} \tag{10}$$

There are two cases to be distinguished regarding the optimal pollution policy. The first is the case of R&D having delivered a backstop technology. The second is the case of no backstop having been invented yet. We start with the first case.

Production using the backstop involves no pollution. Once a technology of the backstop type is present, its shadow price $\mu_i^{Back}(t), i = n$ is therefore zero while that for all polluting technologies ($i = 1, ..., n-1$) is strictly negative once they have accumulated positive stock $S_i(t) > 0$. Hence, $\sigma_n(t) > \sigma_i(t)$ for all $i = 1, ..., n-1$ with $S_i(t) > 0$. It follows from (10) that output of all technologies of the boomerang type is zero unless their stock is zero. If indeed $S_i(t) = 0$, for some $i \in \{1, ..., n-1\}$ there will be an infinitely small time interval $[t + dt]$ during which these polluting technologies will be employed. Otherwise, the backstop produces at full capacity since there are no costs attached to production such that

$$q_i^{Back}(t) = 0 \quad , \forall t \in [t_n, \infty) \quad , i = 1, ..., n-1 \tag{11}$$

$$q_j^{Back}(t) = 1 \quad , \forall t \in [t_n, \infty) \tag{12}$$

with superscript *Back* denoting outputs after a backstop has been invented.

As a result, the stocks of polluting technologies then decrease according to

$$S_i(t) = S_i(t_n)e^{-\delta(t-t_n)} \quad , \forall t \in [t_n, \infty) \,. \tag{13}$$

Once the backstop is developed, the present setting provides no reason for further R&D.

Pollution policy in the second case (only technologies of the boomerang type available) is more complex. There are three relevant cases to be considered: (a) the singular case where all pollution stocks will be symmetric, (b) a non-singular case where one technology has initially a zero stock while stocks of other technologies are at the same positive level and (c) a non-singular case where one technology has initially a zero pollution stock and there are different positive stock levels. This selection is exhaustive because new technologies always start with a zero pollution stock. Case (a) describes the case before the first innovation and after convergence of new and incumbent technologies. If innovation occurs while the economy is in an (a) phase, case (b) is relevant. However, if the economy is in phase (b) or (c) when innovation occurs, (c) is appropriate.

(a) The Singular Solution

Technologies for which the switching function (10) is zero

$$\sigma_i(t) = 0 \tag{14}$$

are on a singular path. Observe that the switching function is zero for more than one technology only if their stocks are symmetric. The following shadow price dynamics apply to all technologies on a singular path

$$\mu_i(t) = \frac{e^{-rt}}{\alpha}\left(\kappa_n(t) - 1\right) \tag{15}$$

$$\dot{\mu}_i(t) = -\frac{e^{-rt}}{\alpha}\left[r\left(\kappa_n(t) - 1\right) - \dot{\kappa}_n(t)\right]. \tag{16}$$

Three relevant cases are considered:[3]

Case 1: $\kappa_n = 0$ and $\dot{\kappa}_n = 0$

Case 2: $\kappa_n > 0$ and $\dot{\kappa}_n = 0$

Case 3: $\kappa_n > 0$ and $\dot{\kappa}_n \neq 0$

Case 1

Here, supply falls short of the fixed unit demand and the constraint (3) is not binding ($\kappa_n = 0$). Using the first order condition (9) and the shadow price dynamics (15) and (16) one gets

$$S_i(t) = \frac{r + \delta}{\alpha d} \tag{17}$$

$$q_i^{Boom}(t) = \frac{\delta(r + \delta)}{\alpha^2 d} \tag{18}$$

with the superscript *Boom* denoting output levels when no backstop is available. This is a steady state that is "incomplete" in the sense that the marginal damage of pollution outweighs the marginal benefit of production before the capacity constraint becomes binding. A higher discount rate, lower persistence of pollution, lower emission intensity and lower marginal damages increase output and stock levels of the incomplete steady state. Both, equilibrium output and

[3]These are the relevant cases because κ_n can not become negative in this problem.

pollution stock of technologies are independent of the number of technologies. However, the condition for this steady state to exist

$$n\frac{\delta(r+\delta)}{\alpha^2 d} \leq 1 \tag{19}$$

is a function of n. For each set of exogenous parameters thus, there is an upper bound of n above which the incomplete steady state is not feasible.

Case 2

This is the complete steady state as the demand constraint (3) is binding ($\kappa_n > 0$) while the corresponding shadow price is constant ($\dot{\kappa}_n = 0$). Again, using (9), (15) and (16) one gets by imposing symmetry

$$S_i(t) = \frac{\alpha}{\delta n} \tag{20}$$

$$q_i^{Boom}(t) = \frac{1}{n}. \tag{21}$$

Equilibrium output is completely determined by the number of available technologies. The steady state pollution stocks are a function of the pollution intensity α, the depreciation rate of pollution δ and the number of technologies. The discount rate r and the slope of the damage function d do not affect the steady state. The complete steady state is feasible if and only if

$$n\frac{\delta(r+\delta)}{\alpha^2 d} > 1 \tag{22}$$

holds. Note, that (19) and (22) are mutually exclusive and exhaustive.

Case 3

Supply is at full capacity ($\kappa_n > 0$) but the shadow price of a marginal increase of the production capacity is changing. Case 3 is therefore not a steady state. From (9), (15) and (16) one gets by imposing symmetry

$$S_i(t) = \frac{\alpha}{\delta n} - \frac{\alpha}{\delta n} e^{-\delta t} \tag{23}$$

$$q_i^{Boom}(t) = \frac{1}{n} \tag{24}$$

This is the most rapid approach path to a steady state when all technologies have equal initial pollution stocks. In $t = 0$ the economy has to be in this case because by assumption $n(0) = 1$.[4] As stocks accumulate according to (23), the economy either reaches the incomplete steady state (Case 1) or approaches the complete steady state (Case 2). Conditions (19) and (22) determine which steady state is relevant.

(b) Innovation with Symmetric Stocks

So far only situations where all technologies have the same pollution stock were analyzed. However, if innovation of k boomerang technologies occurs at some point in time $t_n > 0$ this is no longer the case. While the incumbent technologies $\{1, ..., n-k\}$ have already accumulated some stock, that of new ones $\{n-k, ..., n\}$ is still zero. Hence, pollution stocks and their respective shadow prices differ across new and established technologies. Here we will assume that this is the first innovation at some strictly positive point in time. However, it will be shown later, that the analysis also applies to all subsequent sequences

[4]The same holds for $n(0) > 1$. Since for all $i \in \{1, ..., n(0)\}$ it holds that $S_i(0) = 0$.

of innovation. When k boomerangs are developed the pollution stocks are

$$S_i(t_n) = \frac{\alpha}{\delta(n-k)} - \frac{\alpha}{\delta(n-k)} e^{-\delta t_n} \quad , i = 1, ..., n-k \tag{25}$$

$$S_j(t_n) = 0 \quad , j = n-k+1, ..., n. \tag{26}$$

Here, the singular condition (14) cannot hold for all technologies simultaneously but only for one of the two sets of technologies. Since $S_i(t_n) > S_j(t_n)$ and therefore $\mu_i(t_n) < \mu_j(t_n)$ it has to hold that $\sigma_i(t_n) < \sigma_j(t_n)$. Due to (3), (14) can only hold for the k new technologies while for all $n-k$ old technologies $\sigma_i(t_n) < 0$ and hence

$$q_i^{Boom}(t) = 0 \quad , \forall t \in \left[t_n, \hat{t}_n\right] \quad , i = 1, ..., n-1 \tag{27}$$

$$q_j^{Boom}(t) = \frac{1}{k} \quad , \forall t \in \left[t_n, \hat{t}_n\right] \tag{28}$$

This is the most rapid approach path to a situation where pollution stocks of all technologies are equal. The corresponding stock dynamics are

$$S_i(t) = S_i(t_n) e^{-\delta(t-t_n)} \quad , \forall t \in \left[t_n, \hat{t}_n\right] \tag{29}$$

$$S_j(t) = \frac{\alpha}{\delta k} - \frac{\alpha}{\delta k} e^{-\delta(t-t_n)} \quad , \forall t \in \left[t_n, \hat{t}_n\right] \tag{30}$$

where $\hat{t}_n$ is the point in time where $S_i(\hat{t}_n) = S_j(\hat{t}_n)$. Using (29) and (30) the point of convergence is at

$$\hat{t}_n = t_n + \frac{1}{\delta} \ln \left[\frac{\delta k}{\alpha} S_i(t_n) + 1\right]. \tag{31}$$

From $\hat{t}_n$ until the next innovation all technologies are used at equal shares and stocks grow according to the following 'Case 3'-process

$$S_l(t) = \frac{\alpha}{\delta n} - \frac{\alpha}{\delta n} e^{-\delta(t-\bar{t}_n)} \quad , t > \hat{t}_n, l = 1, ..., n. \tag{32}$$

The virtual starting point of this process $\bar{t}_n$ is determined by

$$S_l(\hat{t}_n) = S_i(\hat{t}_n) \quad , i = 1, ..., n-1, l = 1, ..., n \tag{33}$$

which yields

$$\bar{t}_n = 0. \tag{34}$$

Hence, the path of the pollution stock after innovation and convergence (32) is exactly the same as the one were all n technologies are available at $t = 0$ (23). Subsequent arrivals of boomerang technologies can therefore be analyzed by exactly the same procedure substituting in the respective new values for n and k. This, however, hinges on the condition that innovation occurs after convergence has occurred. The alternative case is analyzed in (c) below.

(c) Innovation with Asymmetric Stocks

Assume a boomerang technology arrives at $t_n \in \{t_{n-1}, \hat{t}_{n-1}\}$ where pollution stocks of technologies $\{1, ..., n-k\}$ have not yet converged. Again, it is optimal to follow the most rapid approach path, i.e.

$$q_i^{Boom}(t) = 0 \quad , \forall t \in \left[t_n, \hat{t}_n\right], i = 1, ..., n-k \tag{35}$$

$$q_n^{Boom}(t) = \frac{1}{k} \quad , \forall t \in \left[t_n, \hat{t}_n\right], i = n-k+1, ..., n. \tag{36}$$

Using a procedure analogous to that used to derive $\hat{t}_n$, the point in time where the stocks of technologies $n-k$ and $\{n-k+1, ..., n\}$ converge is determined as

$$\hat{t}_n = t_n + \frac{1}{\delta} \ln \left[\frac{\delta k}{\alpha} S_i(t_n) + 1\right] \tag{37}$$

Whether or not this case ever arises depends on the optimal timing of R&D. This is analyzed in the next section. Note, that the optimal pollution policy after the development of a backstop technology (11) and (12) is not affected by asymmetric stocks.

4 The Optimal Timing of R&D

4.1 Setup of the Optimal Timing Decision for R&D

The previous section derived the optimal contingent pollution policies. Given these policies, the social planner faces the problem at which points in time to invest into R&D and thereby acquire a new technology that can turn out to be either of the backstop or the boomerang type.

The following analysis is based on recent results on multi-stage dynamic optimization techniques derived by Makris (2001) and Tomiyama (1985). The application of the technique to the problem at hand is natural: Here, a *stage* is defined by reference to the number n of technologies available for production. Switching between stages n and $n+1$ involves carrying out R&D at cost R. More than one switch can occur at any given point in time $t \geq 0$, if optimal. While the necessary conditions derived by Tomiyama (1985) and Makris (2001) are established in the context of a deterministic setting, they are easily modified for the simple discrete probability distribution studied here in order to account for the uncertainty regarding the type of technology developed at the point of switching.

Given the initial endowment of $n(0) = 1$ technologies the optimization problem is as follows

$$\begin{aligned}
\max_{\{t_2,t_3,\ldots,t_N\},\{N\}} &= \int_0^{t_2} e^{-rt}\left[\left(q_1^{Boom} - \frac{d}{2}S_1^2\right)\right]dt - e^{-rt_2}R \\
&+ (1-p)\left\{\int_{t_2}^{t_3} e^{-rt}\left[\sum_{i=1}^{2}\left(q_i^{Boom} - \frac{d}{2}S_i^2\right)\right]dt - e^{-rt_3}R\right\} \\
&+ p\left\{\int_{t_2}^{\infty} e^{-rt}\left[\sum_{i=1}^{2}\left(q_i^{Back} - \frac{d_i}{2}S_i^2\right)\right]dt\right\} \\
&+ \ldots \\
&+ (1-p)^{N-1}\int_{t_N}^{\infty} e^{-rt}\left[\sum_{i=1}^{N}\left(q_i^{Boom} - \frac{d}{2}S_i^2\right)\right]dt \\
&+ p(1-p)^{N-2}\int_{t_N}^{\infty} e^{-rt}\left[\sum_{i=1}^{N}\left(q_i^{Back} - \frac{d_i}{2}S_i^2\right)\right]dt \qquad (38)
\end{aligned}$$

subject to (1), (3) and (7). This is equivalent to (6) with the exception that the optimal pollution policy has already been solved and that the path probabilities (see Figure 1) have been multiplied out. The corresponding Hamiltonian for each stage, where n technologies already exist, is

$$H_n \equiv \sum_{i=1}^{n}\left[e^{-rt}\left(q_i^* - \frac{d}{2}S_i^2\right) + \mu_i(\alpha q_i^* - \delta S_i)\right] \quad , n = 1, ..., N \qquad (39)$$

where the optimal q_i^* is conditional both on the number and type of existing technologies (see Section 3). Given the optimal pollution policies, the applicable necessary conditions for the optimal switching point are essentially those provided by Tomiyama (1985) and Makris (2001), modified however for a setting of two possible outcomes. Two conditions then determine the optimal instant t_{n+1}^* to undertake R&D in order to develop the $n + 1^{st}$ technology. The first condition is a *matching condition* that requires that - in expected terms - the pollution shadow prices of existing technologies are not affected by innovation, i.e.

$$\mu_i^{Boom}(t_{n+1}^*) = E\left(\check{\mu}_i^*(t_{n+1}^*)\right) \quad , i = 1, ..., n. \tag{40}$$

where $\mu_i^{Boom}(t_{n+1}^*)$ is the shadow price of stock i at t_{n+1}^* with n boomerang technologies while $E\left(\check{\mu}_i^*(t_{n+1}^*)\right) = p\check{\mu}_i^{Back}(t_{n+1}^*) + (1-p)\check{\mu}_i^{Boom}(t_{n+1}^*)$ is the expected shadow price of the same stock at the switching instant but 'after' innovation given that optimal pollution policies are implemented. The shadow prices of pollution stocks depend on the optimal pollution policy. Since the latter is conditional on the type of technology developed, so are the shadow prices $\check{\mu}_i^{Back}$ for the case a backstop arrives and $\check{\mu}_i^{Boom}$ when the new technology is a boomerang. Hence the matching condition of Tomiyama (1985) and Makris (2001) for the deterministic case ($\mu_i = \check{\mu}_i$) must hold in expected terms.

The second condition is the *research arbitrage condition*

$$\sum_{n=2}^{N} \left\{ \left[H_n^*\left(t_{n+1}^*\right) + e^{-rt_{n+1}^*} rR - E\left[H_{n+1}^*\left(t_{n+1}^*\right)\right] \right] \delta t_{n+1} \right\} \leq 0 \tag{41}$$

for any admissible perturbation δt_{n+1} in the innovation time t_{n+1}^*. Asterisks indicate optimal values. Again, the value of the optimal post-innovation Hamiltonian $H_{n+1}^*\left(t_{n+1}^*\right)$ depends on the type of technology developed and is therefore represented by its expected value in the research arbitrage condition.

Using both necessary conditions and substituting in the optimal pollution policies this yields (proof see appendix)

$$rR \leq \alpha \left[E\left(\check{\mu}_{n+k}^*\left(t_{n+1}^*\right)\right) - E\left(\check{\mu}_n^*\left(t_{n+1}^*\right)\right) \right] e^{rt_{n+1}^*} \quad , t_{n+1}^* = 0. \tag{42}$$

$$rR = \alpha \left[E\left(\check{\mu}_{n+k}^*\left(t_{n+1}^*\right)\right) - E\left(\check{\mu}_{n+k-1}^*\left(t_{n+1}^*\right)\right) \right] e^{rt_{n+1}^*} \quad , t_{n+1}^* > 0. \tag{43}$$

for the k^{th} additional technology developed at instant t^*_{n+1}. The optimal time to innovate is when the marginal gain of waiting (the left hand sides) is not higher than the expected marginal cost of doing so (the right hand sides). The latter is determined by the difference between the expected shadow price of the new technology ($E\left(\check{\mu}^*_{n+k}\right)$) and that of the lowest pollution stock of an active technology ($E\left(\check{\mu}^*_n\right)$ or $E\left(\check{\mu}^*_{n+k-1}\right)$).

4.2 Characterization of the Optimal Innovation Policy

Here we present and prove the key results on the optimal innovation policy. The emphasis is on developing the essential heuristic steps for characterizing the optimal policy, with some of the algebraic manipulation relegated to the appendix where indicated.

Proposition 1

There is no upfront innovation at the beginning of the planning period ($t = 0$).

Proof. At $t = 0$, the existing as well as any newly developed technology have - by definition - a pollution stock of $S_i(0) = 0$. If research produces a boomerang technology it is perfectly symmetric to any already existing one. Hence, the shadow prices are the same in this case: $\check{\mu}^{Boom}_1(0) = \check{\mu}^{Boom}_{1+k}(0)$. If research produces a backstop technology instead, the shadow price of the perfectly clean technology is zero ($\check{\mu}^{Back}_{n+k}(0) = 0$). The shadow price of any polluting technology at the instant a backstop arrives is given by (see appendix)

$$\breve{\mu}_i^{Back}\left(t_{n+1}^*\right) = -\frac{d}{r+2\delta} S_i\left(t_{n+1}^*\right) e^{-rt_{n+1}^*}. \quad , i = 1, ..., n \tag{44}$$

At the beginning of the planning horizon all pollution stocks are zero and hence $\breve{\mu}_i^{Back}(0) = 0$. Hence, the expected shadow prices of both the initially freely available and any newly developed technology at $t = 0$ are the same: $E\left(\breve{\mu}_{n+k}^*(0)\right) = E\left(\breve{\mu}_n^*(0)\right)$. Plugging this into (42) yields that there is no research upfront if R&D is costly ($R > 0$) and the social planner not infinitely patient ($r > 0$). □

Proposition 2

Innovation is sequential. At most one technology is developed at any point in time.

Proof. If more than one technology is developed ($k > 1$) only the expected shadow prices of new technologies enter condition (43). Pollution stocks for both are zero. Reasoning along identical lines as in the proof for Proposition 1, we obtain $E\left(\breve{\mu}_{n+k}^*\left(t_{n+1}^*\right)\right) = E\left(\breve{\mu}_{n+k-1}^*\left(t_{n+1}^*\right)\right)$. Incorporating this into (43) yields that research is sequential unless R&D is for free ($R = 0$) or the social planner infinitely patient ($r = 0$). □

More detail about the optimal timing of research is obtained by replacing the expected shadow prices in (43) with more explicit terms. First, rewrite (43) using $\breve{\mu}_{n+1}^{Back} = 0$ and $k = 1$ (Proposition 2) as follows

$$rR = \alpha \left\{ (1-p)\breve{\mu}_{n+1}^{Boom}\left(t_{n+1}^*\right) - \left[p\breve{\mu}_n^{Back}\left(t_{n+1}^*\right) + (1-p)\breve{\mu}_n^{Boom}\left(t_{n+1}^*\right)\right] \right\} e^{rt_{n+1}^*} \quad , t_{n+1}^* > 0. \tag{45}$$

$\breve{\mu}_n^{Back}\left(t_{n+1}^*\right)$ is given by (44). Note that there is a link between $\breve{\mu}_{n+1}^{Boom}\left(t_{n+1}^*\right)$ and $\breve{\mu}_n^{Boom}\left(t_{n+1}^*\right)$: Assuming the stocks of both boomerang technologies converge at some point in time (this assumption is shown to be correct in Proposition 3), technologies are at that point perfectly symmetric with respect to their exogenous parameters, stocks and optimal future pollution policies. Hence, at the point of convergence shadow prices of both technologies are the same. Using this link, it is possible to express one shadow price in terms of the other. Given the optimality of most rapid convergence except in the case of further innovations occurring in the meantime (see (27) and (28)), the relation is as follows (proof see appendix)

$$\breve{\mu}_{n+1}^{Boom}(t_{n+1}^*) = \breve{\mu}_n^{Boom}\left(t_{n+1}^*\right) + de^{-rt_{n+1}^*}\left\{\frac{S_n^*\left(t_{n+1}^*\right)}{r+2\delta} - \frac{\alpha}{(r+\delta)(r+2\delta)}\left[1 - \left(\frac{\delta}{\alpha}S_n^*\left(t_{n+1}^*\right)+1\right)^{-\frac{r+\delta}{\delta}}\right]\right\}. \tag{46}$$

Substituting (44) and (46) into (45) yields the *research trigger condition*

$$rR = \frac{\alpha d}{r+2\delta}S_n^*\left(t_{n+1}^*\right) - \frac{(1-p)\alpha^2 d}{(r+\delta)(r+2\delta)}\left[1 - \left(\frac{\delta}{\alpha}S_n^*\left(t_{n+1}^*\right)+1\right)^{-\frac{r+\delta}{\delta}}\right]. \tag{47}$$

This determines the optimal switching times $t_1^*, ..., t_N^*$ and thereby the optimal number of technologies N if innovation occurs only when the pollution stocks of all existing technologies have converged. Hence, the next issue is to proof that this is indeed the case.

Proposition 3

Innovation occurs only at instances at which all available technologies are used simultaneously.

Proof. For any given interval $[t^1, t^2]$ during which no innovation occurs, the gains from innovation are monotonically increasing in the stock of the most recent technology and hence in time. Note that Proposition 2 states that at the instant a technology is developed the gains of further innovation are zero. As the pollution stock of the most recent technology accumulates, gains from innovation increase. The costs of research, on the other hand, are constant. The single crossing property of this setting determines the research trigger condition (47) as the unique optimal switching point. (47) requires all existing technologies to be used simultaneously. Innovation during convergence is therefore ruled out. □

(47) therefore fully characterizes the optimal R&D sequence in this stylized model. Together with the optimal pollution policies derived in Section 3 the optimal joint pollution and R&D program is determined. A specific representation for the corresponding evolution of pollution stocks is given in Figure 2. It depicts a situation with $N = 3$ where - by construction - no backstop arrives. While the actual equilibrium stocks are represented by bold lines, the fine (solid) lines are the approach paths to the steady states given 1, 2 or 3 technologies, respectively. Note that since the capacity constraint is always binding and tech-

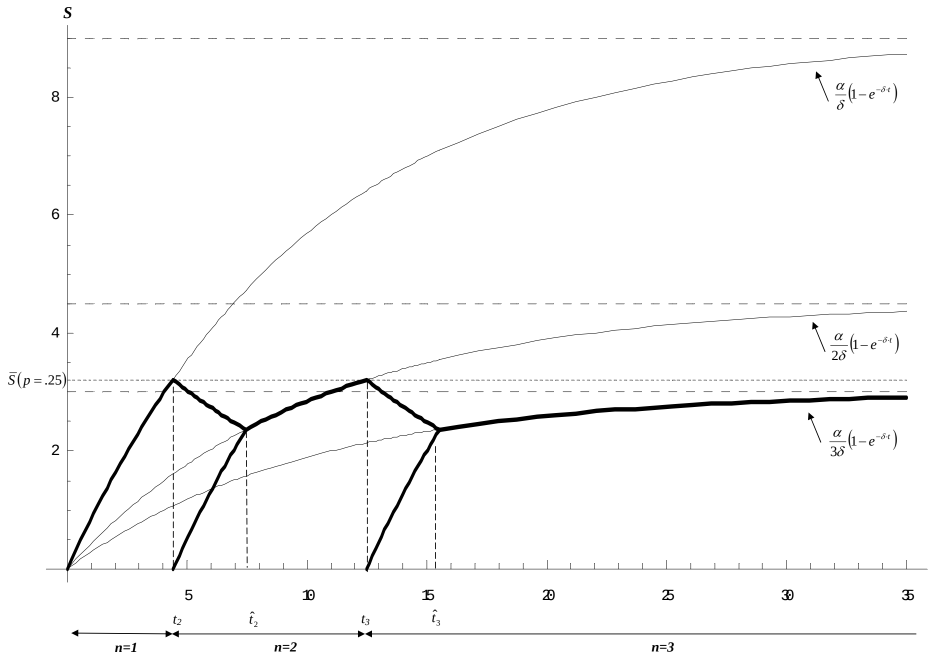

Figure 2: Optimal evolution of stock and R&D sequence (*N=3*) when R&D fails to develop a backstop ($p = 0.25$).

nologies of the boomerang type are symmetric, the approach path to the steady state given one technology is active is also the evolution of the total aggregate pollution stock. Due to (2) this is not proportional to aggregate damages in the economy. The dashed horizontal lines indicate the (hypothetical) steady state levels for $n = 1$, $n = 2$ and $n = 3$. Based on (47), we can say more about the exact link between pollution (stocks) and R&D.

Proposition 4

In the optimum innovation occurs whenever the pollution stock of any technology reaches a constant threshold level $\bar{S}$.

Proof. Time enters the research trigger condition (47) only via the pollution stock of the most recent technology. With all other variables in (47) exogenous parameters, research is triggered each time $S_n^*\left(t_{n+1}^*\right) = \bar{S}$. Moreover, since all pollution stocks are symmetric in all switching instants (Proposition 3) this is equivalent to any pollution stock reaching the trigger level $\bar{S}$. □

The dotted horizontal line in Figure 2 indicates this pollution threshold level. Having established this tight relation between pollution stocks and the timing of innovation we are now in a position to state some further properties of the optimal R&D and pollution trajectories. One important feature is the optimal procedure if R&D (repeatedly) fails to deliver the desired backstop technology. The question here is whether research is carried out - potentially *ad infinitum* - until a backstop is developed or whether R&D eventually ceases even if the pollution problem has not been solved.

Proposition 5

The optimal R&D program has an endogenous stopping point. For any set of parameters with $R > 0$ and $r > 0$, at most $N = \min[\hat{N}, M]$ technologies are developed. $\hat{N}$ is independent of the maximum number of feasible technologies, M.

Proof. Innovation ceases if a backstop technology is developed. If no backstop arrives (either because $p = 0$ or because of bad luck) there is an upper bound on the number of boomerang technologies developed in the optimum. To see this, recall that the steady state pollution stock (20) is strictly decreasing in the number of available technologies n. Moreover, $\lim_{n\to\infty} \frac{\alpha}{\delta n} = 0$. Hence, there is a number of boomerang technologies $\hat{N}$ for which the condition $\frac{\alpha}{\delta \hat{N}} < \bar{S} \leq \frac{\alpha}{\delta(\hat{N}-1)}$ holds. Once the $\hat{N}^{th}$ boomerang is developed, the innovation trigger level will not be reached again. Given that $M \geq \hat{N}$ and since p is independent of both M and n, the size of the set of feasible technologies, M, does not affect the maximum number of technologies, N, developed in an optimal R&D program. □

The optimal stopping rule for R&D is therefore as follows: no further R&D is carried out if either a backstop has been developed or $N \min[\hat{N}, M]$ boomerang technologies have been developed. R&D stops even though a backstop may not have been developed and even though there are still potential technological solutions to be discovered. This pattern of R&D timing has repercussions on the optimal evolution of pollution stocks.

Proposition 6

If the optimal R&D policy requires at least one innovation and M is not binding, pollution stocks overshoot.

Proof. Each time innovation occurs all pollution stocks are at $\bar{S}$ (Propositions 3 and 4). If a backstop is developed, pollution stocks will fall and approach zero in the long run. This is a trivial form of overshooting. If no backstop is developed, then the economy has N boomerang technologies in the long run (Proposition 5). If $M \geq \hat{N}$, the corresponding steady state level of pollution stocks is below the innovation trigger level, each time innovation occurs pollution stocks of all available technologies are above their long run steady state level. Overshooting occurs whether a backstop arrives in the future or not. However, if it is never optimal to undertake R&D, i.e. if $\frac{\alpha}{\delta} \leq \bar{S}$, the pollution stock of the only available technology never exceeds its long run steady state. The same holds if $M < \hat{N}$ and the sequence of innovations stops because the set of potential ideas to solve the pollution problem is exhausted. In this case the long run steady state is above the innovation trigger level, but no R&D occurs because the economy is short of new ideas. □

Hence, even if there is a specific long run pollution target (say for the carbon dioxide concentration in the atmosphere), it can be optimal to exceed this level for some (repeated) periods of time.[5] Moreover, both the periods when stocks overshoot as well as the time between two such periods increases in the number of available boomerang technologies.

Proposition 7

The time between successive innovations is increasing in the number of already

[5]Note that this model abstracts from irreversible catastrophic damages triggered at specific stock levels.

available technologies.

Proof. After a new technology is developed pollution stocks converge. This process takes $\hat{t}_{n+1} - t_{n+1}$. According to (31) the length of this period is independent of the number of technologies already available. The next innovation is triggered if all pollution stocks simultaneously reach $\bar{S}$ again. Since after convergence is completed all technologies are used at a rate of $1/(n+1)$, which is decreasing in n, the time that passes between successive innovations increases in n. □

Although there is no upfront innovation (Proposition 1) the R&D program is front loaded in a sense that the 'density' of innovations, i.e. the number of innovations within a given but sufficiently large interval of time, is decreasing in time.

So far the probability of a backstop to arrive by virtue of R&D did not affect the validity of any of the previous propositions. However, it is an important determinant of the optimal timing of research.

Proposition 8

The maximum number of technologies developed, N, is weakly increasing in the probability, p, that a backstop is developed by R&D. The time between successive innovations is strictly decreasing in p.

Proof. Total differentiating (47) yields

$$\frac{d\bar{S}}{dp} = -\frac{\alpha}{r+\delta} \cdot \frac{1 - \left[\frac{\delta}{\alpha}\bar{S} + 1\right]^{-\frac{r+\delta}{\delta}}}{1 - (1-p)\left[\frac{\delta}{\alpha}\bar{S} + 1\right]^{-\frac{r+2\delta}{\delta}}} < 0. \tag{48}$$

The pollution stock threshold $\bar{S}$ is decreasing in p. However, N is weakly decreasing in $\bar{S}$ (see proof of Proposition 5). In addition, the time between successive innovations is increasing in $\bar{S}$ (see Figure 2). Both the time interval pollution stocks require to converge (see (31)) and the time interval spent rebuilding pollution stocks back to $\bar{S}$ are reduced. □

The intuition is straightforward. A backstop technology is always more desirable than a technology of the boomerang type. Increasing the probability that research produces a backstop while keeping the costs of R&D, R, constant, makes research more attractive. It is carried out earlier and potentially more often. Note, however, that in contrast to the maximum number the expected number of innovations can decrease in p. Since research ceases as soon as a backstop is developed, which becomes more likely, it becomes less likely that the technology portfolio actually reaches its upper bound N.

Figure 3 illustrates the relation between the probability that research produces a clean backstop and the maximum size of the technology portfolio, N, if M is not binding. The two bold horizontal lines represent the threshold pollution stock $\bar{S}$ for $p = 0$ and $p = 1$, respectively. The range in between covers all feasible threshold levels corresponding to specific probabilities to develop a backstop. Note that the relation between p and $\bar{S}$ is concave (see also (48)). A marginal increase of p results in a larger decrease in the threshold if p is small than if it is large. The dots are steady state pollution stocks for a given number

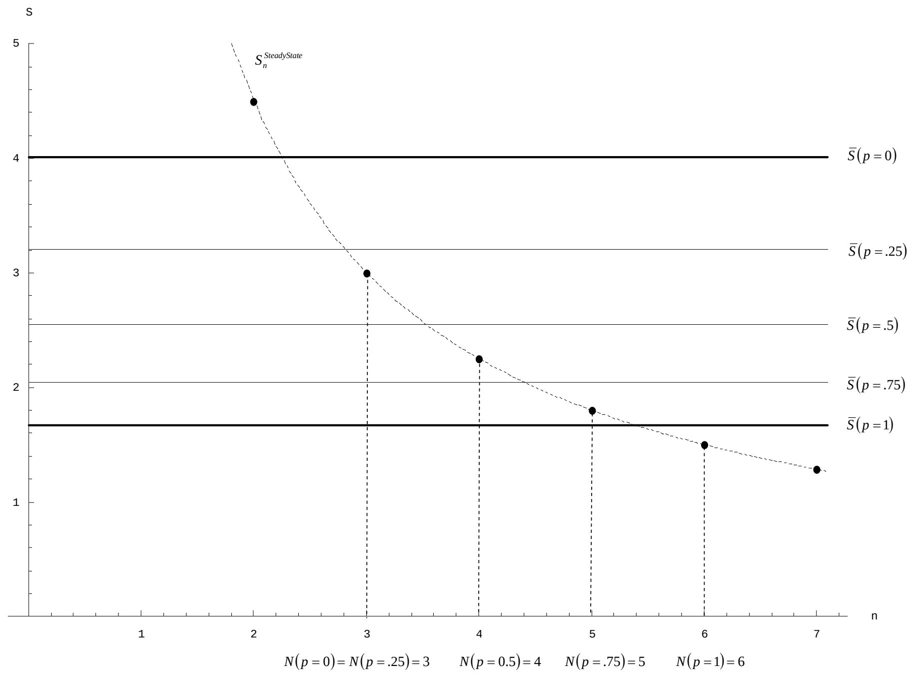

Figure 3: The upper bound on the technology portfolio.

of active technologies, n. All dots reside on the dotted hyperbolic line that represents $S_n^{SteadyState} = \frac{\alpha}{\delta n}$ if n is not restricted to natural numbers. However, since the number of technologies is always a natural number and M might be binding, the upper bound to the technology portfolio, N, is only weakly increasing in p. In Figure 3 this occurs, e.g. when increasing p from zero to 0.25 (the latter appears also in Figure 2). In both cases $N = 3$ since it is the largest $S_n^{SteadyState}$ that is below the respective $\bar{S}(p)$.

5 Conclusion

In much of the literature on environmental R&D, it is common to assume that the outcome of the next (or most recent) R&D effort will be a backstop technology that resolves the intertemporal constraints of the environmental problem forever. This is a productive modeling shortcut that has enabled important results on the optimal timing of R&D to be derived under very general conditions. However, its premise is empirically at least arguable, as we illustrate with two prominent examples. In this paper, we consider a situation in which the next R&D effort generates two possible types of technology, either a backstop technology or another polluting technology (referred to as a 'boomerang'). The type of technology generated is only revealed after R&D expenditure has been incurred. We analyze the impact of this technological uncertainty on the optimal R&D and pollution policy for a policymaker faced with stock pollution and costly R&D. We develop a simple and tractable model in which we apply and extend recent results on the necessary conditions of multi-stage optimal control problems to our problem. This allows an intuitive and natural representation of the discrete nature of technological change that we want to capture here. We also provide a small, but novel extension of the theory to simple discrete probability distributions over possible stages based on the policymaker's beliefs about the relative likelihood of a backstop or a 'boomerang'.

The paper provides a full characterization of the optimal policy in the context of the model. Given the optimal pollution policy, the degree of technological uncertainty does not affect the fundamental structure of the optimal R&D policy, which is strictly sequential and has an endogenous stopping point. However, the timing of innovations and the maximum size of the technology portfolio are affected: To the extent that invention of a backstop becomes less likely, R&D is carried out later and the maximum number of technologies is smaller. The lower productivity of R&D in expected terms spills over into environmental policy in the form of higher equilibrium pollution stocks.

The properties of the optimal policy depend technically on the assumptions about the welfare function, the nature of the pollution problem, the capacity constraint in output, and the specific characterization of R&D. Some qualifications are therefore in order. First, generalizations of the welfare function (5) and relaxation of the output constraint will give rise to additional motives for carrying out R&D for reasons that are well known from the literature on R&D, such as cost savings and more benign environmental characteristics etc. Overall R&D effort will be higher therefore, but for reasons unrelated to technological uncertainty. To the extent that capital stock effects are relevant, costs or benefits are no longer linear in output and the pollution policy will evolve more smoothly. However, the effects on the optimal R&D policy are unlikely to affect its fundamental character that is grounded in the evolution of the pollution stock. Generalized pollution dynamics, on the other hand (see e.g. Tahvonen and Salo (1996)), would lead in some cases to ambiguous effects on the optimal policy choice. It is generalizations of this type that are important areas for future research.

A Appendix

A.1 Combining the necessary conditions: (40) and (41) to (42) and (43)

Condition (41) requires that $G(t_{n+1}^*) = H_n^* \left(t_{n+1}^*\right) + e^{-rt_{n+1}^*} rR - E\left[H_{n+1}^*\left(t_{n+1}^*\right)\right]$ is non-negative for all $\delta t_{n+1} < 0$ and non-positive for all $\delta t_{n+1} > 0$. Otherwise, there exist perturbations for which (41) becomes positive. $G(t_{n+1}^*) = 0$ is therefore a necessary condition for all $t_{n+1}^* > 0$. For $t_{n+1}^* = 0$, G is allowed to be negative. First consider innovation at some $t_{n+1}^* > 0$, where

$$H_n^* \left(t_{n+1}^*\right) + e^{-rt_{n+1}^*} rR = E\left[H_{n+1}^*\left(t_{n+1}^*\right)\right] \tag{A.1}$$

is a necessary condition. Substituting (39) into (A.1) yields

$$\begin{aligned}
e^{-rt_{n+1}^*}\left[\sum_{i=1}^{n+k-1}\left(q_i^{Boom} - \frac{d}{2}S_i^{*2}\right) + rR\right] + \sum_{i=1}^{n} \mu_i^{Boom}\left(\alpha q_i^{Boom} - \delta S_i^*\right) \quad = \\
(1-p)\left\{e^{-rt_{n+1}^*}\sum_{i=1}^{n+k}\left(\check{q}_i^{Boom} - \frac{d}{2}\check{S}_i^{*2}\right) + \sum_{i=1}^{n+k}\check{\mu}_i^{Boom}\left(\alpha \check{q}_i^{Boom} - \delta \check{S}_i^*\right)\right\} \\
p\left\{e^{-rt_{n+1}^*}\sum_{i=1}^{n+k}\left(\check{q}_i^{Back} - \frac{d}{2}\check{S}_i^{*2}\right) + \sum_{i=1}^{n+k}\check{\mu}_i^{Back}\left(\alpha \check{q}_i^{Back} - \delta \check{S}_i^*\right)\right\}
\end{aligned} \tag{A.2}$$

where a ˇ indicates post-innovation values and the superscripts *Back* and *Boom* optimal values in case the new technology is a backstop or a boomerang, respectively. Using $\sum_{i=1}^{n} q_i^{Boom} = \sum_{i=1}^{n+k} \check{q}_i^{Boom} = \sum_{i=1}^{n+k} \check{q}_i^{Back} = 1$, $S_j(t_{n+1}^*) = 0$ for all $j = n+1, ..., n+k$ and the optimal pollution policy in case a backstop arrives (11) and (12) this reduces to

$$e^{-rt^*_{n+1}}\left[1-\sum_{i=1}^{n}\frac{d}{2}S_i^{*2}+rR\right]+\sum_{i=1}^{n+k-1}\mu_i^{Boom}\left(\alpha q_i^{Boom}-\delta S_i^*\right)= \tag{A.3}$$

$$(1-p)\qquad\left\{e^{-rt^*_{n+1}}\left[1-\sum_{i=1}^{n}\frac{d}{2}\check{S}_i^{*2}\right]+\sum_{i=1}^{n}\check{\mu}_i^{Boom}\left(\alpha\check{q}_i^{Boom}-\delta\check{S}_i^*\right)+\sum_{i=n+1}^{n+k}\alpha\check{\mu}_j^{Boom}\check{q}_j^{Boom}\right\}$$

$$p\qquad\left\{e^{-rt^*_{n+1}}\left[1-\sum_{i=1}^{n}\frac{d}{2}\check{S}_i^{*2}\right]-\sum_{i=1}^{n}\check{\mu}_i^{Back}\delta\check{S}_i^*+\alpha\check{\mu}_{n+k}^{Back}\right\}.$$

Using the pollution shadow prices' matching condition (40), a straightforward stock matching condition $S_i^*=\check{S}_i^*$ for all $i\in\{1,n\}$ and the absence of a stock constraint for the backstop $\check{\mu}_{n+k}^{Back}=0$, the optimal pollution policy in case a boomerang is developed (27) and (28) and rearranging terms yields

$$rR=\alpha\left[\frac{(1-p)}{k}\sum_{j=n+1}^{n+k}\check{\mu}_j^{Boom}-\sum_{i=1}^{n+k-1}\mu_i^{Boom}q_i^{Boom}\right]e^{rt^*_{n+1}}. \tag{A.4}$$

Note that by symmetry $\check{\mu}_{n+1}^{Boom}=...=\check{\mu}_{n+k}^{Boom}$ and that $\sum_{i=1}^{n+k-1}\mu_i^{Boom}q_i^{Boom}=\mu_i^{Boom}q_{n+k-1}^{Boom}$ for all optimal pollution policies. Using (40) again, (A.4) simplifies to (43).

The proof for $t^*_{n+1}=0$ works analogously and yields (42).

A.2 Shadow Prices when a Backstop Arrives: (44)

If a backstop arrives at t^*_{n+1} the stock of all polluting technologies deteriorates according to (13). Using (9) yields

$$\check{\mu}_i^{Back}(t)=e^{\delta(t-t^*_{n+1})}\left[\check{\mu}_i^{Back}+\frac{de^{2\delta t^*_{n+1}}}{r+2\delta}S_i(t^*_{n+1})\left(e^{-(r+2\delta)t^*_{n+1}}-e^{-(r+2\delta)t}\right)\right] \tag{A.5}$$

The transversality condition (7) requires that the limit for $t\to\infty$ of the optimal Hamiltonian with the final technology portfolio is zero. Substituting (A.5) and the optimal pollution policy (11) and (12) into (7) yields (44).

A.3 Shadow Price of a New Technology at $t^*_{n+1} > 0$: (46)

During convergence after the arrival of a new boomerang technology at t^*_{n+1}, (29) and (30) describe the evolution of stocks for technologies n and $n+1$. Using (see (9)) one gets the following shadow price dynamics

$$\begin{aligned} \breve{\mu}_n^{Boom}(t) &= e^{\delta(t-t^*_{n+1})} \\ &\quad \cdot \left[\breve{\mu}_n^{Boom}\left(t^*_{n+1}\right) + dS^*_n\left(t^*_{n+1}\right) e^{2\delta t^*_{n+1}} \left(\frac{e^{-(r+2\delta)t^*_{n+1}} - e^{-(r+2\delta)t}}{r+2\delta} \right) \right] \quad \text{(A.6)} \\ \breve{\mu}_{n+k}^{Boom}(t) &= e^{\delta(t-t^*_{n+1})} \left[\breve{\mu}_{n+k}^{Boom}\left(t^*_{n+1}\right) + \frac{\alpha d}{\delta k} e^{\delta t^*_{n+1}} \right. \quad \text{(A.7)} \\ &\quad \left. \cdot \left(\frac{e^{-(r+\delta)t^*_{n+1}} - e^{-(r+\delta)t}}{r+\delta} - \frac{e^{-(r+\delta)t^*_{n+1}} - e^{-(r+2\delta)t+\delta t^*_{n+1}}}{r+2\delta} \right) \right] \end{aligned}$$

At $\hat{t}_{n+1}$ stocks and hence the shadow prices of incumbent and new technologies converge. Hence, from $\breve{\mu}_n^{Boom}\left(\hat{t}^*_{n+1}\right) = \breve{\mu}_{n+k}^{Boom}\left(\hat{t}^*_{n+1}\right)$ and (A.6), (A.7) and (31) it follows that

$$\begin{aligned} \breve{\mu}_{n+k}^{Boom}(t^*_{n+1}) &= \breve{\mu}_n^{Boom}\left(t^*_{n+1}\right) \quad \text{(A.8)} \\ &\quad + \frac{d}{r+2\delta} e^{-rt^*_{n+1}} \left[S^*_n\left(t^*_{n+1}\right) + \frac{\alpha}{\delta k} \right] \left[1 - \left(\frac{\delta k}{\alpha} S^*_n\left(t^*_{n+1}\right) + 1 \right)^{-\frac{r+2\delta}{\delta}} \right] \\ &\quad - \frac{\alpha d}{\delta k(r+\delta)} e^{-rt^*_{n+1}} \left[1 - \left(\frac{\delta k}{\alpha} S^*_n\left(t^*_{n+1}\right) + 1 \right)^{-\frac{r+\delta}{\delta}} \right]. \end{aligned}$$

Further simplifying yields (46).

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