University of Heidelberg

Department of Economics

Discussion Paper Series No. 425

Task Difficulty, Performance Measure Characteristics and the Trade-Off between Insurance and Well-Allocated Effort

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May 2006

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May 8, 2006

Abstract

When designing incentives for a manager, the trade-off between insurance and a "good" allocation of effort across various tasks is often identified with a trade-off between the responsiveness (sensitivity, precision, signal-noise ratio) of the performance measure and its similarity (congruity, congruence) to the benefit of the manager's employer. A necessary condition for the trade-off between responsiveness and similarity to be meaningful is that a perfectly congruent measure creates a higher benefit than an equally responsive non-congruent measure. We show that this condition is met if and only if all tasks are exactly equally difficult and there are no spill-overs or synergies across tasks. This means that for most practical purposes, notions of responsiveness and similarity are not informative about the tradeoff between insurance and allocation. In order to understand this trade-off, task difficulty has also to be taken into account.

JEL-Codes: M41, M52, J33, D82

Keywords: hidden action, multitasking, incentives

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*This is an improved version of the paper entitled "On the Prudence of Rewarding for A while Hoping for B" which circulated as IZA DP-765, CMPO DP 03/067, and BGSE DP 7-2003.

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1 Introduction

The activity of a manager who carries out various tasks is often reflected in a rich set of accounting and other data. If the employer of the manager wants to link the manager's pay to his performance, such multi-dimensional information needs to be aggregated into a single performance measure as forcefully argued by Jensen et al. (2004). There are different ways of aggregating the information leading to different performance measures. Some of these measures may be better than others. But which measure should the employer use?

Before addressing this question, we want to examine what the employer knows. Typically, she has a fair idea about the influence of the manager on the variation of the performance measure. In other words, she knows how *responsive* the measure is. Specific concepts which formalize this notion is the sensitivity-precision concept (Banker and Datar, 1989), the signal-noise ratio (Kim and Suh, 1991), the mean-preserving spread of the likelihood (Kim, 1995) and the risk-minimization component (Feltham and Wu, 2000). Another piece of information that is often available to the employer is how well the measure reflects her benefit from the activity. The idea of similarity between measure and benefit is embodied in the concept of congruity (Feltham and Xie, 1994), discongruity (Datar et al., 2001), distortion (Baker, 2000), congruence (Baker, 2002) and non-congruency (Feltham and Wu, 2000). Here, we are interested in how useful performance measure characteristics like responsiveness and similarity are when deciding what measure to use.

Suppose there are two performance measures that aggregate the available data differently: the first measure perfectly reflects the benefit of the employer; it is *congruent* with the benefit while the second is not. Both are equally affected by factors beyond the control of the manager; they are equally *responsive*. Is the congruent measure preferable to the non-congruent one? Here, we show that the answer is "no" for most cases of practical relevance.

There are two effects which determine the usefulness of a performance measure: how much effort can be bought without imposing too much uncertainty on the manager (an insurance problem) and how this effort is allocated across tasks (an allocative problem). The insurance problem is often thought to be related to the responsiveness of the measure and the allocative problem to its similarity with the benefit (Feltham and Xie 1994, Baker 2000, Feltham and Wu 2000, Baker 2002). If responsiveness indeed reflects how much effort can be bought and similarity how well this effort is allocated, then a measure should generate higher surplus if it is more responsive. Likewise, more similarity with the benefit should lead to a larger surplus. In particular, a congruent measure should be preferable to an equally responsive non-congruent measure. This superiority of a congruent measure is thus necessary for performance measure characteristics to reflect insurance and allocative considerations.

It is a priori not clear why the trade-off between insurance and allocation can be reflected by characteristics of the performance measure. Neither this nor the superiority of congruent measures over equally responsive non-congruent measures has been proven.[1] In a single task model, Kim and Suh (1991) show that a larger signal-noise ratio (more responsiveness) indicates lower agency costs (higher surplus) and Kim (1995) obtains a similar result for the mean-preserving spread. If the manager fully controls the variation of the measure (maximal responsiveness), Datar et al. (2001) find that measures with lower discongruity (higher similarity) lead to a larger surplus. So, in absence of multiple tasks more responsiveness is better and in absence of noise, more similarity is preferable. Unfortunately, these results do not help us to answer what happens if there are multiple tasks and the measure is noisy at the same time; the proofs by Kim and Suh (1991), Kim (1995), and Datar et al. (2001) do not extend to this case. Moreover, Datar et al. (2001) demonstrate with an example that care has to be taken when generalizing from the single-task to the multiple-task model:[2] a higher sensitivity (more responsiveness) may actually decrease the attractiveness of a congruent performance measure. [3]

Another relevant question is whether performance measure characteristics like responsiveness and similarity provide enough information. Can we

[1]It has not even been formalized what a "good" allocation of a given effort is.

[2]See also the distinction between congruent and incongruent sensitivity by Banker and Thevaranjan (2000).

[3]In this example, however, they vary the benefit and sensitivity at the same time and it is hence not possible to isolate the effect of the change in sensitivity.

ignore how difficult the manager finds which task? If the manager is in full control of all relevant variables and these variables are combined to form a congruent performance measure, it is possible to devise an incentive scheme such that the manager fully internalizes the employer's benefit and the efficient solution is reached. This case is an example where task difficulty does not influence the design of optimal incentives. If the manager is not in full control and requires insurance, internalization will be partial. This may lead the manager to neglect costly but beneficial tasks. Should the incentive scheme not correct for this by focusing attention on such tasks? If yes, a performance measure that emphasizes these tasks would be preferable to a congruent measure and task difficulty would influence the optimal design.

Most of the models dealing with multiple task problems and featuring some notion of responsiveness and similarity assume that all tasks are exactly equally difficult. According to Feltham and Wu (2000) who draw on Wu (1995), this is an innocuous assumption as "results are substantially the same".[4] The assumption is sometimes justified on the grounds that the units in which the manager's activity is measured are arbitrary. Rescaling these units (say from hours to minutes), it is possible to obtain a problem with equalized and independent marginal costs. Following this argument and the prevailing practice in the literature, task difficulty does not seem to matter for the trade-off between insurance and allocation. This suggests, it can be ignored when choosing a performance measure.

In order to check whether task difficulty affects the choice of the performance measure, we use the multiple-task principal-agent model by Holmström and Milgrom (1991) in the form popularized by Feltham and Xie (1994) and allow for costs to differ and interact across tasks. We introduce this model in Section 2. There are several notions of similarity between measure and benefit (Feltham and Xie 1994, Baker 2000, Feltham and Wu 2000, Datar, Kulp and Lambert 2001, Baker 2002), which exist for this model. All these notions agree that a measure is congruent with the benefit if the systematic effect of the manager on the measure is a multiple of that on the benefit. In geometrical terms, the vector of marginal effects on the measure and on the benefit have the same direction. Currently, there is only one notion describing responsiveness in this model: the risk minimization compo-

[4]see footnote 4 on page 168 in Feltham and Wu (2000)

nent by Feltham and Wu (2000). This concept has two drawbacks: it requires the employer to know the optimal activity in the absence of contractual constraints (first-best activity) and it does not reduce to the signal-noise ratio of Kim and Suh (1991) when there is only a single task. As an alternative, we build on the ideas of Banker and Datar (1989) and Kim and Suh (1991) and define a multiple-task signal-noise ratio as the (inner vector) product of the marginal effects of the manager on the measure (signal) divided by the variation beyond the control of the manager (noise). This definition is perfectly in line with Baker (2002), who fixes the length of the marginal effect vector and the variance of the noise term in order to hold responsiveness constant.

We begin the analysis, in Section 3, by considering the case where tasks are equally costly and find the conventional wisdom confirmed: a performance measure with a given signal-noise ratio generates the highest surplus only if it is congruent. So, for the specific case of equal costs our question is answered: the congruent performance measure is always preferable. Any trade-off between notions of similarity and responsiveness implicitly takes this result for granted; here, it is proven for the first time. The result crucially depends on the notion of responsiveness: if the risk minimization component by Feltham and Wu (2000) is used instead of the signal-noise ratio, the measure which maximizes surplus is not necessarily congruent.

In Section 4, we leave the restrictive setting of exactly equal costs at all tasks. The second and central finding is that if marginal costs differ, a performance measure with a given signal-noise ratio no longer maximizes surplus when it is congruent. The vector of marginal effects of the optimal performance measure may even be almost orthogonal to that of the benefit. It is irrelevant for this result how responsiveness is precisely defined: the generated surplus always depends on the difficulty of tasks and so does its maximizer when fixing a quantity that does not depend on this difficulty (e.g. the signal-noise ratio as defined here or the risk minimization component by Feltham and Wu 2000). Task difficulty does matter for the choice of the optimal performance measure and hence notions of responsiveness and similarity cannot capture the trade-off between insurance and allocative considerations; the idea that characteristics of a performance measure may be used to determine its consequences on the amount and allocation of effort is thus confined to the rather limited case of identical costs.

The difficulty of tasks has hence to be taken into account when selecting a performance measure. But how? It is not possible to take the original problem with differing costs, transform this problem to the identical-independent cost setting by rescaling effort, and carry out comparisons and trade-offs in the usual fashion because the signal-noise ratio is not invariant to such rescaling. In Section 5, we explore a different way and adjust the notion of responsiveness for costs. We prove that when comparing performance measures with the same cost-adjusted signal-noise ratio, the maximal surplus is generated by a congruent measure, again.

In the final section, we summarize our results, discuss them and examine their implications.

2 The model

Consider the designer of an incentive scheme in a multiple task setting, who wants to choose the best performance measure in order to provide incentives to an agent (e.g. manager) who creates a benefit for a principal (e.g. employer). We follow the literature and use the working horse model of multiple task principal-agent analysis (see Feltham and Xie 1994, Baker 2000, Feltham and Wu 2000, Banker and Thevaranjan 2000, Datar, Kulp, and Lambert 2001, Baker 2002) to relate to this literature and to obtain simple closed form solutions. This model was initially devised by Feltham and Xie (1994) by using the assumptions of the single-task linear-exponential-normal (LEN) model by Spremann (1987) in the multi-task agency problem of Holmström and Milgrom (1991).

In this model, the principal's benefit is a linear function of the agent's effort vector e:

$$B(e, \eta) = \beta' e + \eta,$$

where $\beta' = (\beta_1, \ldots, \beta_n)$ is a vector of coefficients and η is a noise term with mean zero. The noise term reflects influences on the benefit beyond the control of the agent. Likewise, performance measures (which can be regarded as some aggregation of the available information) are a linear function of efforts:

$$P(e, \epsilon) = b' e + \epsilon,$$

where $b' = (b_1, \ldots, b_n)$ is a coefficient vector and ϵ is a normally distributed noise term with expected value zero and variance σ^2. This noise term captures any effects on the performance measure that cannot be influenced by the agent. The noise which influences the benefit may stand in an arbitrary relation to the noise influencing the performance measure: it may be independent, correlated or identical ($\epsilon = \eta$). Hence the description of the performance measure encompasses the special case that it is identical to the benefit ($b = \beta$ and $\epsilon = \eta$).

Next, we link the coefficient vector of the performance measure to that of the benefit.

Definition 1 (Congruence)**.** *Performance measure and benefit are* congruent *if their coefficient vectors have the same direction: $b = \gamma\beta$ for some γ different from zero.*

This definition relates to various congruity concepts: congruent performance measures minimize the discongruity of Feltham and Xie (1994), the non-congruency by Feltham and Wu (2000), the distortion by Baker (2000) and the incongruity of Datar et al. (2001) while they maximize the congruence of Baker (2002). Thus, congruent performance measures are an important benchmark case in all similarity concepts.

It is customary in the multitasking literature to assume that the wage is linear in the performance measure: $W(P(e)) = w_0 + w_1 P(e)$. This simplifying assumption is not innocuous because the optimal incentive scheme in the single-task case is non-linear (Mirrlees, 1999). However, linearity can be justified under certain additional assumptions about the timing of decisions (Holmström and Milgrom 1987 and Hellwig and Schmidt 2002). While the principal is supposed to be risk-neutral, the agent has a negative exponential utility function $U(v) = -e^{(-vr)}$, where $v := W(P(e)) - C(e)$ is the income of the agent and r is the Arrow-Pratt measure for risk aversion. It is usually assumed that costs are quadratic and that marginal costs are identical across tasks (Feltham and Xie 1994, Baker 2000, Banker and Thevaranjan 2000, Feltham and Wu 2000, Datar, Kulp, and Lambert 2001, Baker 2002).[5] Here, we examine the more general case of differing and interrelated marginal costs:

$$C(e) = e'Ce,$$

[5]There are a few (unpublished) exceptions such as Wu (1995) and Ratto (2006).

where the matrix $C = (c_{ij})$ is assumed to be symmetric and non-negative definite, so that costs are always positive: $e'Ce > 0$ if $e \neq (0, \ldots, 0)'$. The more specific cost function normally used can be obtained by setting $C = \frac{1}{2}I$, where I is the identity matrix.

Which concept should we use for responsiveness? We want to rely on the results of Kim and Suh (1991) for the single-task case and hence introduce a generalization of their concept to the multi-task model.

Definition 2 (Signal-noise ratio)**.** *The* signal-noise ratio *is the ratio of the inner product of the expected marginal effect of activity on the measure divided by the variance of the noise of the measure:*

$$\rho := \frac{\left(\frac{\partial E_\epsilon(P(e,\epsilon))}{\partial e}\right)' \left(\frac{\partial E_\epsilon(P(e,\epsilon))}{\partial e}\right)}{\sigma^2} = \frac{b'b}{\sigma^2}.$$

In the single-task case, b is a real number and the signal-noise ratio reduces to that defined by Kim and Suh (1991) and is equivalent with the product of sensitivity and precision considered by Banker and Datar (1989).[6] The definition of the signal-noise ratio reflects the idea that a longer coefficient vector means that the respective performance measure is more susceptible to changes in effort and thus more "informative".[7] On the other hand, fixing the ratio between the squared length and the variance implies that two performance measures with the same signal-noise ratio are somehow "equally blurred by noise". Compare for example two measures, (b, σ^2) and $(\tilde{b}, \tilde{\sigma}^2)$, where the second is similar to the first but twice as sensitive to changes in efforts: $\tilde{b} = 2b$. Then, this higher sensitivity only translates into an advantage if the second measure is less than twice as noisy : $\tilde{\sigma} < 2\sigma$; if it is exactly twice as noisy, the signal-noise ratio is the same and the two measures are equivalent in the sense that the consequences of using one measure can be perfectly replicated by using the other measure and adjusting the wage rate. Note that even if they have the same signal-noise ratio, performance measures may still emphasize different effort dimensions. In other words, the

[6]Banker and Datar (1989) also suggest a a signal-noise ratio for multiple signals. This suggestion does not concern us here, as we are supposing that the information of various signals is already embodied in the performance measure.

[7]Presumably, this is the reason why Baker (2002) fixes this length to illustrate his idea of congruence.

allocative properties are not yet determined.

After having introduced congruence and signal-noise ratio as characteristics of the performance measure, we want to return to the agency problem. What is the maximal surplus that principal and agent can obtain when they cannot contract on effort? For the answer, we follow the standard procedure (see e.g. Macho-Stadler and Pérez-Castrillo 1997) of assigning the role of the mechanism designer to the principal and let her receive any benefit from the operation of the incentive scheme. An incentive constraint represents the rationality of the effort choice of the agent. Finally, a participation constraint ensures that the benefit is indeed generated by the mechanism and not by exploiting the agent; for convenience, we standardize the certainty equivalent of the outside option to zero, so that the respective utility is minus one. Overall, the maximization program takes the form:

$$\max_{w_0,w_1} \mathrm{E}\left(B(e) - w_0 - w_1 P(e)\right) \tag{1}$$

$$\text{such that} \quad e \in \mathrm{argmax}_{\tilde{e}} \mathrm{E}\left(U\left(w_0 + w_1 P(\tilde{e}) - C(\tilde{e})\right)\right) \tag{2}$$

$$\text{and} \quad \mathrm{E}\left(U\left(w_0 + w_1 P(e) - C(e)\right)\right) \geq -1. \tag{3}$$

Thanks to the model assumptions, we can solve this maximization program and find the optimal (linear) incentive scheme for any performance measure P, where it is convenient to parameterize this performance measure in b and the signal-noise ratio ρ rather than in b and σ^2. Note, that the two parameterizations are completely equivalent.

Lemma 1. *Given a performance measure with weights b and signal-noise ratio ρ, the optimal rate of performance pay is:*

$$w_1^* = \frac{\beta' C^{-1} b}{b' C^{-1} b + 2r\frac{b'b}{\rho}}. \tag{4}$$

Under this rate, the agent will exert the effort:

$$e^* = \frac{w_1^*}{2} C^{-1} b, \tag{5}$$

which yields a surplus of:

$$\phi(b, \rho) = \frac{1}{4} \frac{b' C^{-1} \beta \beta' C^{-1} b}{b'(C^{-1} + 2\frac{r}{\rho} I) b}. \tag{6}$$

The proof for this lemma is an extension of the standard proof to more general costs and can be found in Appendix A.

As we expect, a higher Arrow-Pratt measure of risk aversion r or a lower signal-noise ratio ρ reduces the performance pay rate w_1^*. Interestingly, the optimal wage rate also depends on the costs C. Generally, higher costs lead to a lower wage rate but more importantly, the cost matrix enters the relationship between b and β. This implies that the wage rate may change, simply because the relative importance of tasks for the performance measure or the benefit is different. Thus, the relative weights for effort may determine to what degree a performance measures is used.

Lemma 1 allows us to quickly assess the surplus generated by a performance measure characterized by (b, ρ). It thereby enables us to search for the optimal coefficient vector and to check whether it is congruent in the next section.

3 Equal marginal costs across tasks

If the various notions of similarity are informative about a good allocation of effort across tasks, the highest surplus amongst measures with the same signal-noise ratio should be generated by a congruent measure. Formally, we can find the optimal measure by maximizing the surplus (as computed in equation (6)) in b while fixing ρ. This leads to the following observation.

Proposition 1. *Suppose costs for tasks are equal ($C = c \cdot I$, where c is a positive real number). Consider only measures with the same signal-noise ratio ρ. Then, the measure which maximizes the surplus $\phi(b, \rho)$ is congruent.*

The proof of this proposition is less obvious than the expected and simple result suggests (see Appendix B). Trade-offs between insurance and allocation can only be reflected in performance measure characteristics when more similarity is better for given responsiveness. The proposition thus brings partial relief to various models which illustrate such trade-offs using notions of similarity and responsiveness (Feltham and Xie 1994, Feltham and Wu 2000, Baker 2000, Baker 2002). The relief is only partial for two reasons. First, we only consider the benchmark comparison between a congruent and a non-congruent performance measure. If the respective similarity concept is indeed useful, more similarity should also generate a higher surplus when comparing several non-congruent measures with the same signal-noise ratio. Such comparisons have to be carried out using the respective notion of similarity. Introducing all these concepts and comparing them seems a lengthy exercise and due to space constraints, we refrain from doing so. It might, however, prove useful in order to discriminate between the different concepts of similarity. The second reason why the relief is only partial is that not all notions of responsiveness can be employed. If the signal-noise ratio is replaced by the risk-minimization component by Feltham and Wu (2000), the resulting measure is not necessarily congruent (see Appendix C).

4 When costs differ across tasks

As the assumption of equal costs merely simplifies the analysis and as it is still possible to obtain closed form solutions if costs differ and interact, we relax the assumption and move to the more realistic situation that the agent finds working on some task easier relative to other tasks. Although the problem gets more involved, the optimal performance measure can still be computed (the respective details are in Appendix D). It turns out that if cost differences are present, the finding of Proposition 1 is not valid anymore.

Proposition 2. *Given arbitrary costs (e.g. $C \neq c \cdot I$) and considering all performance measures with the same signal-noise ratio ρ, the performance measure that maximizes the surplus takes the form:*

$$b^* = k \left(I + \frac{2r}{\rho} C \right)^{-1} \beta,$$

where k is the following standardization factor: $k = \pm\sqrt{\frac{\rho\sigma^2}{\beta'(2\frac{r}{\rho}C+I)^{-2}\beta}}$.

In fact, the optimal coefficient vectors may be almost orthogonal. Consider a two-dimensional example where the agent has costs $C(e) = \frac{c}{2}e_1^2 + \frac{1}{2}e_2^2$. Using the formula in Proposition 2, we find that the optimal performance measure has the coefficients:

$$b_1^* = k \cdot \frac{\beta_1}{1 + c\frac{r}{\rho}} \text{ and } b_2^* = k \cdot \frac{\beta_2}{1 + \frac{r}{\rho}}.$$

Suppose that the effort on the first task is more costly ($c > 1$). Then, the optimal performance measure puts more emphasis on the cheap effort relative to a congruent measure. The more risk-averse the agent or the lower the signal-noise ratio of the performance measure, the stronger is the focus on the cheap task. Imagine that the bulk of the benefit is created by the first task $\beta_1 = 1 - \epsilon$ whereas the second task only contributes a little: $\beta_2 = \epsilon$. Then, the relative weight of the first dimension in comparison with the second dimension is $\frac{b_1^*}{b_2^*} = \frac{1-\epsilon}{\epsilon} \frac{\rho+cr}{\rho+r}$. This implies that for increasing costs c and decreasing signal-noise ratio ρ, the relative importance of the first effort in the optimal measure shrinks to zero although the relative importance of this effort for benefit creation is rather high. The principal does not only prefer a non-congruent to a congruent performance measure with the same signal-noise ratio but the emphasis on tasks in the measure may have very little to do with their importance for the creation of benefit.

Thus generally, the principal does not choose a congruent measure ($b = \gamma\beta$) but prefers a particular non-congruent measure. If we do not want to restrict the analysis to specific benefit functions,[8] a congruent performance measure only maximizes the surplus if costs are identical across tasks. This result can be derived by asking when the optimal performance measure b^* is a multiple of β ($\beta^* = \gamma\beta$ for some real number γ) in the above proposition; a full analysis is in Appendix E.

Corollary 1. *Given any benefit coefficient vector β and signal-noise ratio ρ, congruent performance measures maximize the surplus if and only if costs are identical and independent along all effort dimensions: $C = I \cdot c$, with $c \in I\!R$.*

This result is valid independently from the specific notion of responsiveness. Any such notion, whether signal-noise ratio, risk minimization component or any other characteristic of the performance measure, is independent of costs. The surplus, however, does depend on costs (see equation (6)) and so does its maximizer under a side-constraint which is independent of costs.

The results in this section imply that unless the designer knows costs to be identical and independent, the trade-off between insurance and allocative

[8] If the benefit coefficient vector β happens to be an eigenvector of the matrix $\left(I + \frac{2r}{\rho}C\right)^{-1}$, it also maximizes the surplus – even if costs are not identical and independent.

considerations is not reflected in performance measure characteristics. Thus, the practical value of using and weighing these characteristics seems rather limited.

5 Incorporating task difficulty

The last section presented the slightly puzzling finding that under different costs, non-congruent and even almost orthogonal performance measures create a higher surplus than congruent performance measures with the same signal-noise ratio. In this section, we will identify the source of this finding and suggest a way to establish the optimality of congruent measures even in the presence of different costs.

There are two possible sources for the puzzling finding. First, congruent performance measures are simply not leading to the optimal allocation of effort across tasks under second-best conditions. Second, the signal-noise ratio is not appropriate to control for the insurance problem. In this section, we argue that the latter explanation is valid.

Recall that the optimal wage-rate, w_1^*, depends on the composition of the performance measure – even if its Euclidean length $\sqrt{b'b}$ is kept constant. However, there is no reason why the wage rate should vary when facing the same insurance problem. The only reason for the wage rate to change is when the performance measure imposes more (or less) uncertainty on the agent. This already hints to the fact that the signal-noise ratio may be insufficient to capture the insurance problem. Further indication comes from the example with the almost orthogonal measure, which was discussed in the previous section. To obtain an almost orthogonal optimal performance measure, we either had to increase risk-aversion or decrease the signal-noise ratio. Both quantities are clearly related to the insurance problem. Using the same example, it can also be illustrated why the signal-noise ratio does not fully capture the insurance problem. Even though the signal-noise ratio is fixed, the mechanism designer can choose a measure which emphasizes a particular task. This does not only imply that the agent focuses on this task, it also means that this task can be better measured: the controlled variation with respect to this task increases relative to the uncontrolled variation (which stays constant). So, while the overall controlled variation, as represented by the length of the performance measure, does not change, the controlled variation along a particular dimension may very well change. Consequently, relatively less uncertainty is imposed on the agent and the performance wage rate can be increased. In essence, a trade-off between allocation and insurance occurs and a "bad" allocation is accepted for a less noisy payment to the agent *despite* our attempt to eliminate such trade-offs by holding the signal-noise ratio constant.

Independent and identical costs represent a knife-edge case, where trade-offs between insurance and allocation do not matter once the signal-noise ratio is fixed. This suggests to adjust the signal-noise ratio for costs in order to re-establish the results from the equal cost case. We want to lay the foundation for such an adjustment by considering a broader class of signal-noise ratios.

Definition 3 (modified signal-noise ratio)**.** *The* modified signal-noise ratio *is the ratio of the squared length of the marginal effect measured using some arbitrary norm M divided by the variance of the noise: $\rho_M := \frac{b'Mb}{\sigma^2}$, where M is some positive definite matrix.*

But which norm M should be chosen? Is there a particular choice which allows us to re-establish optimality of congruent measures? How is this choice related to costs? The following proposition provides the respective answers (proof see Appendix F).

Proposition 3. *Within the class of performance measures with signal-noise ratio ρ_M, congruent measures maximize the surplus if and only if $M = C^{-1}$.*

Thus, we have to use the modified signal-noise ratio $\rho_{C^{-1}} := \frac{b'C^{-1}b}{\sigma^2}$ if we want congruent measures to be optimal. Moreover, even if we do not limit attention to modified signal-noise ratios and consider more general any notion of responsiveness, this notion has to depend on costs if we want to re-establish the optimality of congruent performance measures (see Appendix G).

An alternative way to deal with differing costs seems to be to re-scale effort in such a way that costs become equal and independent. We would hope that we can then rely on Proposition 1 and congruent measures are again attractive. Unfortunately, the signal-noise ratio is not invariant to rescaling.

Thus, even when the problem is transformed, we cannot work with the ordinary signal-noise ratio but have to use a cost-adjusted version of it (see Appendix H).

Cost-adjusting whatever notion of responsiveness we wish to use has a major drawback. It requires parameters that are likely to be private information of the agent. While it is possible in a single task setting to compare the quality of measures by looking alone at its characteristics, we know already that the multiple-task setting also requires information on how benefit is created. In the general case, which we have considered here for the first time, even that is not sufficient: agent's preferences also need to be known. The value of a performance measure in a general multi-task setting is hence highly context specific: it depends on parameters of the principal (in form of β) and on parameters of the agent (in form of C).

6 Conclusion

Is it possible to decide between performance measures on the basis of their characteristics? Insurance and allocative considerations are often identified with the responsiveness (signal-noise ratio, precision, risk minimization component) of a performance measure and its similarity to the employer's benefit (congruity, discongruity, congruence, non-congruence). If this identification is justified, congruent performance measures are better than equally responsive non-congruent measures. Here, we checked whether this simple intuition holds and gained the following insights.

First, if all tasks are exactly equally difficult, congruent measures are more attractive than equally responsive non-congruent ones. This result is often taken for granted; here, we have formalized and proven it for the first time. The result is not valid for all notions of responsiveness: it holds when a multi-dimensional version of the signal-noise ratio is used but not for the risk-minimization component by Feltham and Wu (2000). The result provides a necessary but not a sufficient condition for similarity and responsiveness to be informative criteria when designing incentives. It only tells us that a congruent measure generates the highest benefit and not that a measure with higher similarity to the benefit yields more surplus. The validity of this more general statement depends on the specific definition of similarity. Future research could address the question for which notion of similarity (congruity, discongruity, congruence, non-congruence, etc.) more similarity implies a higher surplus when responsiveness is held constant. This might prove useful to decide which of these notions is most suitable to describe the effect of a measure on the quality of effort allocation.

Second, once the task difficulty is not the same across tasks (marginal costs differ), the measure which generates the largest surplus amongst all measures with the same signal-noise ratio is not congruent. Congruent measures are no longer preferable to equally responsive non-congruent measures; it is no longer possible to compare measures on the basis of their characteristics; and the trade-off between more and better allocated effort can no longer be reflected by notions of responsiveness and similarity. Respective illustrations in the literature are thus confined to the rather special case of independent and identical costs.

Third, the information that is required to find optimal incentives seems to increase in the complexity of the contractual environment. In a single-task model with a risk-averse manager and risk-neutral employer, the signal-noise ratio suffices to rank different measures; details of the preferences of the employer or manager do not matter. It has been an important achievement of the literature initiated by Feltham and Xie (1994) to point out that preferences of the employer are important once there are multiple tasks. It has, however, been neglected that the manager's preferences are equally important. In multiple task models, the value of performance measures depends on who is using the performance measure and *for whom* it is used. Accordingly, the designer needs to be informed about benefits and costs – information that may be difficult to obtain. If the employer designs the scheme, the manager has all reason to misrepresent his costs and as they are multi-dimensional it is hard if not impossible to elicit them with a menu of contracts. If the manager designs the scheme, he often does not know how different tasks influence the employer's benefit. So, overall incentive design in the multiple-task context is based on rather strong informational requirements.

Fourth, in the case that the employer knows the preferences of the manager, we suggest a way to incorporate them. The notion of responsiveness (here the signal-noise ratio) needs to be modified in a specific way. Then, the comparison between congruent and non-congruent measures becomes meaningful again. A seemingly attractive alternative would be to re-scale effort in order to have identical costs across tasks and to carry out comparisons in the transformed problem. This, however, is not possible because the transformation alters the signal-noise ratio. In order to consider the same class of performance measures, the signal-noise ratio needs to be once more adjusted for costs – albeit in a different way. It is hence simpler to directly work with the modification of the signal-noise ratio in the untransformed problem.

Fifth, we identify the reason why a congruent measure does not maximize the surplus amongst all measures with the same signal-noise ratio. As the optimal measure puts more and not less emphasis on cheap tasks, the reason cannot be that the manager should be drawn towards costly but beneficial tasks. Rather, it has to do with a trade-off between insurance and effort allocation that is not reflected by the signal-noise ratio and the similarity of the measure to the benefit. It is possible to increase the influence of one task and lower the influence on others while leaving the signal-noise ratio constant. Such a change improves insurance properties but distorts allocation. For the knife-edge case of equal and independent costs, these two effects exactly outweigh each other. But if costs are lower on some task, allocating effort towards this task leads to smaller losses than before and the insurance outweighs the allocative effect. Again, the specific notion of responsiveness is not crucial. As long as fixing responsiveness does not fully determine the allocation across tasks but leaves some freedom, it will not completely reflect the insurance side of the problem. The explanation why non-congruent measures are optimal thus brings us back to the central theme of this article: the designer of incentives has to take costs into account when trading-off insurance and allocation of effort.

References

Baker, George. 2000. The use of performance measures in incentive contracting. *American Economic Review* **90**(2) 415–420.

Baker, George. 2002. Distortion and risk in optimal incentive contracts. *Journal of Human Resources* **37**(4) 728–751. Special Issue on Designing Incentives to Promote Human Capital.

Banker, R., Srikant Datar. 1989. Sensitivity, precision, and linear aggregation of signals for perfomance evaluation. *Journal of Accounting Research* **27**(1) 21–39.

Banker, Rajiv D., Alex Thevaranjan. 2000. Goal congruence and evaluation of performance measures. Downloaded at http://som.utdallas.edu/aim/Working

Datar, Srikant, Susan Kulp, Richard Lambert. 2001. Balancing performance measures. *Journal of Accounting Research* **39**(1) 75–92.

Feltham, G.A., M. Wu. 2000. Public reports, information acquisition by investors, and management incentives. *Review of Accounting Studies* **5**(2) 155–190.

Feltham, Gerald A., Jim Xie. 1994. Performance measure congruity and diversity in multi-task Principal/Agent relations. *The Accounting Review* **69**(3) 429–453.

Hellwig, Martin F., Klaus M. Schmidt. 2002. Discrete-time approximations of the holmström-milgrom brownian-motion model of intertemporal incentive provision. *Econometrica* **70**(6) 2225–2264.

Holmström, Bengt, Paul Milgrom. 1987. Aggregation and linearity in the provision of intertemporal incentives. *Econometrica* **55**(2) 303–328.

Holmström, Bengt, Paul Milgrom. 1991. Multitask principal-agent-analysis: Incentive contracts, asset ownership, and job design. *Journal of Law, Economics, and Organization* **7** 24–52.

Jensen, Michael C., Kevin J. Murphy, Eric Wruck. 2004. Remuneration: Where we've been, how we got to here, what are the problems, and how to fix them. Finance Working Paper 44, ECGI.

Kim, S.K., Y.S. Suh. 1991. Ranking of accounting information systems for management control. *Journal of Accounting Research* 386–396.

Kim, Son Ku. 1995. Efficiency of information system in an agency model. *Econometrica* **63**(1) 89–102.

Macho-Stadler, Ines, D. Perez-Castrillo. 1997. *An Introduction to the Economics of Information: Incentives and Contracts*. Oxford University Press, Oxford.

Mirrlees, James A. 1999. The theory of moral hazard and unobservable behavior (part 1). *Review of Economic Studies* **66** 3–21. Reprint of a manuscript from 1975.

Ratto, Marisa. 2006. Task-specific effort costs and the trade-off between risk and efficiency. Working Paper 06/143, Center for Market and Public Organisation, Bristol.

Salanié, Bernard. 1998. *Economics of contracts*. MIT Press, Massachussets.

Spremann, Klaus. 1987. Agent and principal. Gunter Bamberg, Klaus Spremann, eds., *Agency Theory Information and Incentives*. Springer, New York, Berlin, London, and Tokyo, 3–37.

Wu, M. G. H. 1995. Public report, private information, managerial compensation and effort allocation. Ph.D. thesis, University of British Columbia.

A Proof for Lemma 1

Using the linearity of the performance measure, the normality distribution of the noise, and the shape of the utility function, the base wage can be chosen to compensate the agent for any harmful effects of uncertainty ($w_0 = C(e) - w_1 b'e + w_1^2 r \frac{b'b}{\rho}$), so that the participation constraint can be eliminated from the program (see e.g. Salanié 1998). Recalling the definitions of performance measure, benefit, and costs, as well as the risk neutrality of the principal, the program then simplifies to:

$$\begin{aligned} &\max_{w_1} \beta' e - e'Ce - \frac{w_1^2}{2} r \frac{b'b}{\rho} \\ &\text{such that } e \in \operatorname{argmax}_{\tilde{e}} \mathrm{E}\left(U\left(w_0 + w_1 b'\tilde{e} - \tilde{e}'C\tilde{e}\right)\right). \end{aligned} \tag{7}$$

Consider the effort choice problem of the agent for a given incentive scheme (w_0, w_1) (side constraint). Because the utility function is monotonic in the received wage minus the effort costs, the agent chooses effort so as to maximize this difference:

$$\max_e w_1 b'e - e'Ce.$$

The objective function is concave because the second derivative is a symmetrical, negative definite matrix $(-C)$. Thus, the maximizer can be determined by the first-order condition. Solving for e yields:

$$e = \frac{w_1}{2} C^{-1} b. \tag{8}$$

Note, that C^{-1} exists because C is positive definite. Replacing e in equation (7) by the optimal effort from equation (8) yields the following expression for the objective function:

$$\max_{w_1} \frac{w_1}{2} \beta' C^{-1} b - \frac{w_1^2}{2^2} b' C^{-1} b - \frac{w_1^2}{2} r \frac{b'b}{\rho}. \tag{9}$$

Again, the objective function is concave and solving the first order condition gives the maximizer:

$$w_1 = \frac{\beta' C^{-1} b}{b' C^{-1} b + 2r \frac{b'b}{\rho}} \tag{10}$$

Using the optimal wage rate in (9) finally results in the surplus from an optimal incentive scheme based on the performance measure (b, ρ):

$$\phi(b, \rho) = \frac{1}{4} \frac{b' C^{-1} \beta \beta' C^{-1} b}{b'(C^{-1} + 2\frac{r}{\rho} I) b}.$$

B Proof of Proposition 1

Several steps in the proof are also needed to prove other results in this article. We summarize these steps in the following lemma.

Lemma 2. *The set of maximizers of the expression*

$$\frac{b' C^{-1} \beta \beta' C^{-1} b}{b' H b},$$

where H is a symmetric and positive definite matrix, is

$$\left\{ b^* \middle| b^* = k \cdot (CH)^{-1} \beta, \text{ with } k \in I\!R \right\}.$$

Proof. Proof. Because H is symmetric and positive definite, we can decompose it: $H = P\Lambda P'$ and define $H^{\frac{1}{2}} = P\Lambda^{\frac{1}{2}}P'$. Now, define $\tilde{b} := H^{\frac{1}{2}}b$, so that $b = H^{-\frac{1}{2}}\tilde{b}$ and consider the transformed problem:

$$\max_{\tilde{b}} \frac{\tilde{b}'H^{-\frac{1}{2}}C^{-1}\beta\beta'C^{-1}H^{-\frac{1}{2}}\tilde{b}}{\tilde{b}'\tilde{b}}.$$

To advance on this problem, we fix the length of $\tilde{b}$ to some arbitrary value k^2: $\tilde{b}'\tilde{b} = k^2$. By varying k, we will later obtain the set of all possible solutions. The respective Lagrangian for a given k is:

$$L(\tilde{b}, k) = \tilde{b}'H^{-\frac{1}{2}}C^{-1}\beta\beta'C^{-1}H^{-\frac{1}{2}}\tilde{b} - \lambda\left(\tilde{b}'\tilde{b} - k^2\right).$$

The corresponding first-order conditions are:

$$\left((H^{-\frac{1}{2}})'C^{-1}\beta\beta'C^{-1}H^{-\frac{1}{2}} - \lambda I\right)\tilde{b}^* = 0 \text{ and } (\tilde{b}^*)'\tilde{b}^* = k^2. \tag{11}$$

The first condition is an eigenvalue problem; to obtain $\tilde{b}^*$, we have to find the eigenvalues λ of the matrix $(H^{-\frac{1}{2}})'C^{-1}\beta\beta'C^{-1}H^{-\frac{1}{2}}$. By defining $x := (H^{-\frac{1}{2}})'C^{-1}\beta$, the matrix can be re-written as xx' and it becomes apparent that the matrix is symmetric and of rank one. Due to the latter, there can only be one non-zero eigenvalue. Hence, this value is identical to the trace of the matrix xx':

$$\mathrm{EV}(xx') = \mathrm{tr}(xx') = \mathrm{tr}(x'x) = x'x,$$

where $\mathrm{EV}(\cdot)$ denotes the eigenvalue and $\mathrm{tr}(\cdot)$ is the trace-operator. Replacing the eigenvalue in the eigenvalue problem, we get:

$$(xx' - x'xI)\tilde{b}^* = 0.$$

Obviously, $\tilde{b}^* = kx$ is a solution to this problem. Again due to the rank of xx', it is also the only solution. To recover the solution in the original problem, we have to reverse the transformation:

$$b^* = H^{-\frac{1}{2}}\tilde{b}^* = kH^{-\frac{1}{2}}x = kH^{-\frac{1}{2}}H^{-\frac{1}{2}}C^{-1}\beta = kH^{-1}C^{-1}\beta = k(CH)^{-1}\beta$$

The set of all solutions is then obtained by letting k vary. $\square$

We can now employ this lemma to find the maximizer of the surplus for a given signal-noise ratio. In order to do so, we set $C = I$ and $H = I(1 + \frac{2r}{\rho})$. Then, the quantity in the lemma becomes the surplus and this surplus is maximized for all elements of the set

$$\left\{ b^* \middle| b^* = k \cdot \left(1 + \frac{2r}{\rho}\right)^{-1} \beta, \text{ with } k \in I\!R \right\}.$$

A subset of this set can be obtained by setting $k := \left(1 + \frac{2r}{\rho}\right) \cdot \frac{\sigma \cdot \sqrt{\rho}}{\sqrt{\beta'\beta}}$. The elements of this subset still maximize the surplus but they also fulfill the constraint of having signal-noise ratio ρ. Moreover, they are multiples of the vector β and hence congruent performance measures. Consequently, congruent performance measures maximize the surplus for a given signal-noise ratio.

C Risk minimization component

Feltham and Wu (2000) suggest to measure the noise resulting from a performance measure by a *risk minimization component*, which is defined in equation (8) in their article. In the case of a single performance measure considered here and in our notation, this component is:

$$S := r \cdot \frac{b'}{\sigma} e^{FB}, \tag{12}$$

where e^{FB} is the effort vector in the first-best situation, i.e., when effort e can be legally enforced. This effort equals the benefit coefficient vector $e^{FB} = \beta$ (see page 160 in Feltham and Wu 2000). Overall the risk minimization component is

$$S = \frac{r}{\sigma} \cdot b'\beta.$$

The surplus generated by a performance measure in the case considered by Feltham and Wu ($C = \frac{1}{2}I$) is:

$$\Phi(b, \rho) = \frac{1}{2} \frac{b'\beta\beta'b}{b'(I + \sigma^2 r I)b} = \frac{1}{2} \frac{b'\beta\beta'b}{b'b(1 + \sigma^2 r)}.$$

Using the definition of S, this is equivalent to

$$\Phi(b, \rho) = \frac{1}{2} \frac{S^2}{b'b(1 + \sigma^2 r)} \cdot \frac{\sigma^2}{r^2} = \frac{1}{2} \cdot \frac{1}{(\frac{r^2}{\sigma^2} + r^3)} \cdot \frac{S^2}{b'b}.$$

Holding S constant, the surplus no longer depends on β and consequently the maximizer of the surplus will not depend on β. Accordingly, the optimal performance measure is independent of the benefit of the principal. Thus there are no restrictions placed on the shape of the performance measure in relation to the benefit and it does not matter for surplus maximization whether it is congruent or not.

D Proof of Proposition 2

The proof of this result relies again on Lemma 2 which we stated and proved in Appendix B The matrix $\left(C^{-1} + \frac{2r}{\rho} I\right)$ is symmetric and positive definite. Thus, we can apply Lemma 2 with $H = \left(C^{-1} + \frac{2r}{\rho} I\right)$ and solve the unrestricted problem of maximizing $\phi(b, \rho)$. The set of maximizers for the unrestricted problems is

$$\left\{ b \,\middle|\, b = k \left(I + \frac{2r}{\rho} C \right)^{-1} \beta \text{ with } k \in \mathbb{R} \right\}.$$

Not all these maximizers have the required signal-noise ratio of ρ. However, all performance measure vectors with

$$\left\{ b \,\middle|\, b = k \left(I + \frac{2r}{\rho} C \right)^{-1} \beta \text{ with } k = \pm \sqrt{\frac{\rho \sigma^2}{\beta' (2\frac{r}{\rho} C + I)^{-2} \beta}} \right\}$$

are also maximizers on the restricted set and are hence a solution to the restricted maximization program.

E Proof of Corollary 1

For a congruent performance measure to maximize surplus, it must hold that $b^* = \gamma \beta$ for any β. By Proposition 2, this is equivalent to $(2\frac{r}{\rho} C + I) = I\kappa$, where κ is any real number (The choice of κ is not limited because the standardization factor k neutralizes any choice). Thus, congruent performance measures maximize surplus if and only if $C = I \frac{\kappa - 1}{2} \frac{\rho}{r}$ for any κ. In other words, costs must be a multiple of the identity matrix: $C = cI$, where c is some positive real number $c := \frac{\kappa - 1}{2} \frac{\rho}{r}$. Note that $c > 0$ because C is positive definite by definition.

F Proof of Proposition 3

We express the surplus in terms of the modified signal-noise ratio, so that H becomes $H = (C^{-1} + M\frac{2r}{\rho_M})$. Applying once more Lemma 2 from Appendix B, we get the following set of maximizers:

$$\left\{ b^* \middle| b^* = k \cdot \left(I + CM\frac{2r}{\rho_M} \right)^{-1} \beta, \text{ with } k \in I\!R \right\}.$$

Thus, by a similar argument as in the proof for Corollary 1, b^* will only be a multiple of β if $M = C^{-1}$.

G Impossibility to eliminate agent's costs

Is there any way to control for noise while ignoring agent's costs such that congruent performance measures maximize surplus?

Recall from Section H that the surplus generated by a performance measure is:

$$\frac{1}{4}\frac{\tilde{b}'\tilde{\beta}\tilde{\beta}'\tilde{b}}{\tilde{b}'(I + 2RC)\tilde{b}},$$

where R is some component capturing the uncertainty properties, e.g. $R = r\sigma^2$ or $R = r\frac{r}{\rho}$. By Lemma 2 with $H =' (I + 2RC)$, the performance measure maximizing this surplus is

$$\tilde{b}^* = k\left(C(I + RC)\right)^{-1}\tilde{\beta}.$$

Congruence would require that $\tilde{\beta} = k\left(C(I + RC)\right)^{-1}\tilde{\beta}$. This equation is never going to hold for arbitrary $\tilde{\beta}$ unless R depends on C.

H Rescaling

Can we transform a problem with arbitrary costs to a problem with identical and independent costs and do trade-offs and comparisons in this simple world? The answer is no. The problem of finding the optimal performance measure while responsiveness is fixed is not invariant to rescaling because rescaling affects the responsiveness of a performance measure.

Lemma 3. *If we re-scale efforts in a problem with arbitrary costs $C \neq I$ such that costs become identical and independent, the signal-noise ratio does not stay constant: the problem of finding the optimal measure amongst all equally responsive measures is not invariant to rescaling.*

Proof. Proof. If we want to find some linear transformation $\tilde{e} = Te$ such that

$$C(e) = e'Ce = \tilde{e}'\tilde{e} = \tilde{C}(\tilde{e}),$$

we have to use $T = C^{\frac{1}{2}}e$, where $C^{\frac{1}{2}}$ is some decomposition of the positive definite matrix C such that $(C^{\frac{1}{2}})'C^{\frac{1}{2}} = C$. Using this transformation, we get

$$\begin{aligned} B(e) = &\quad \beta' e + \epsilon = \beta' C^{-\frac{1}{2}}\tilde{e} + \epsilon &= \tilde{B}(\tilde{e}) \\ P(e) = &\quad b'e + \epsilon = b'C^{-\frac{1}{2}}\tilde{e} + \epsilon &= \tilde{P}(\tilde{e}), \end{aligned}$$

and the benefit coefficient vectors in the transformed problem is $\tilde{\beta} := (C^{-\frac{1}{2}})'\beta$ while the measure coefficient vector becomes $\tilde{b} := (C^{-\frac{1}{2}})'b$. The signal-noise ratio in the transformed problem is $\frac{\tilde{b}'\tilde{b}}{\sigma^2} = \frac{b'C^{-1}b}{\sigma^2}$ while the signal-noise in the original problem was $\frac{b'b}{\sigma^2}$. Since $C \neq I$, the signal-noise ratio thus changes. Consequently, the side-condition in the maximization problem changes and hence the maximization problem itself is not invariant to rescaling. □

In order to ensure that the same problem is solved before and after rescaling, the signal-noise ratio has to be rescaled, too. Only if we consider measures with $\frac{\tilde{b}'C\tilde{b}}{\sigma^2}$ after rescaling, we are examining the same set of measures as before rescaling: $\frac{\tilde{b}'C\tilde{b}}{\sigma^2} = \frac{b'b}{\sigma^2}$. Using this altered signal-noise ratio, we find that congruent performance measures maximize surplus. So, we cannot just take the problem with differing costs, transform the problem to identical costs, and do the usual trade-offs in the familiar setting but we have to adjust the signal-noise ratio for costs. Note that this adjustment is different from the adjustment in Proposition 3.