University of Heidelberg

Department of Economics

Discussion Paper Series | No. 424

Brown-von Neumann-Nash Dynamics: The Continous Strategy Case

Josef Hofbauer, Jörg Oechssler
and Frank Riedel

December 2005

Brown-von Neumann-Nash Dynamics: The Continuous Strategy Case*

Josef Hofbauer
Department of Mathematics
University College London

Jörg Oechssler
Department of Economics
University of Heidelberg

Frank Riedel
Department of Economics
University of Bonn

December 14, 2005

Abstract

In John Nash's proofs for the existence of (Nash) equilibria based on Brouwer's theorem, an iteration mapping is used. A continuous–time analogue of the same mapping has been studied even earlier by Brown and von Neumann. This differential equation has recently been suggested as a plausible boundedly rational learning process in games. In the current paper we study this Brown–von Neumann–Nash dynamics for the case of continuous strategy spaces. We show that for continuous payoff functions, the set of rest points of the dynamics coincides with the set of Nash equilibria of the underlying game. We also study the asymptotic stability properties of rest points. While strict Nash equilibria may be unstable, we identify sufficient conditions for local and global asymptotic stability which use concepts developed in evolutionary game theory.

Journal of Economic Literature Classification Numbers: C70, 72.

Key words: learning in games, evolutionary stability, BNN.

*We would like to thank seminar audiences at Berkeley, Stanford, UC Santa Barbara, Madison, Zürich, UCL, at the Econometric Society World Congress in London, and in particular Vince Crawford, Ken Judd, and Bill Sandholm for comments and discussions. We acknowledge financial support from the Austrian Science fund (P 15281), UCL's Centre for Economic Learning and Social Evolution (ELSE), and the SFB/TR 15 by the Deutsche Forschungsgemeinschaft.

1 Introduction

In his famous existence proof of (Nash) equilibria, John Nash [23] used the following iteration mapping to show the existence of a fixed point with the help of Brouwer's theorem[1]

$$p'_s = \frac{p_s + [u(e_s, p) - u(p, p)]_+}{1 + \sum_{s' \in S}[u(e_{s'}, p) - u(p, p)]_+}, \tag{1}$$

where $[a]_+ := \max[a, 0]$. The continuous–time analogue of the same mapping has been studied even earlier by Brown and von Neumann [7] as a novel method for proving the existence of (and calculating) the value in zero–sum games. For the general, non–zero–sum case, it reads as follows:

$$\dot{p}_s = [u(e_s, p) - u(p, p)]_+ - p_s \sum_{s' \in S}[u(e_{s'}, p) - u(p, p)]_+ \tag{2}$$

This differential equation has recently been suggested as a plausible boundedly rational learning process in games (see e.g. Skyrms [31], Swinkels [32], Weibull [34], Berger and Hofbauer [1, 2], Hofbauer [20], Meertens et al. [22] and Sandholm [26, 27, 28]). In honor of its three inventors it has been named *Brown–von Neumann–Nash dynamics* (BNN). In these papers several useful results on the (asymptotic) stability of Nash equilibria with respect to the BNN dynamics have been derived in finite normal form games. The current paper seeks to extend those results to the case of continuous strategy spaces. This is important since for many games of interest (oligopolies, public goods, war of attrition) the strategies (prices, quantities, timing) are best modelled as continuous.[2]

Selection dynamics like the replicator dynamics from evolutionary biology (see e.g. [33], [19]) do not allow to introduce new strategies into the population. A strategy may be superior but if it was not present in the

[1] We present here the version for symmetric two person games with the usual notation: p is a mixture on the set S of pure strategies; p_s denotes the probability of $s \in S$, e_s is the degenerate strategy with probability 1 on s, and $u(\cdot, \cdot)$ is the mixed extension of the payoff function.

[2] Other important applications with continuous strategy spaces are the evolution of preferences (see e.g. Güth and Yaari [15], or Heifetz et al. [18]) and games of incomplete information (see e.g. Ely and Sanholm [11]).

initial population, it can never be used in the future. For the case of human strategic interaction this seems too restrictive. In contrast, the BNN dynamics satisfy the property of "inventiveness" (Weibull [34]), or equivalently, "noncomplacency" (Sandholm [26]). Namely, if there are any (used or unused) strategies that (would) perform better than the current population average, at least one of them must increase in frequency. In particular, new strategies can enter if they yield better than average payoffs.

To be specific let us assume that players are more likely to adopt a new strategy the greater the payoff difference between the new strategy and the current population average. This careful, conservative rule is reminiscent of the proportional imitation rule of Schlag [29] who shows that his "proportional imitation rule" has certain optimality properties in simple decision problems. It is also closely related to the regret–matching rule of Hart and Mas–Colell [16], [17] where better strategies are also adopted with a probability that is proportional to the apparent gain the new strategy yields, only that gains are calculated using the regret one has for not having played the new strategy right from the start.

The remainder of this paper is structured as follows. In the next section we show that the BNN dynamics are well defined for very general strategy spaces. This is not obvious as we have to consider a differential equation on a measure space. Furthermore, we show that under some mild assumptions on the strategy space and for bounded and Lipschitz continuous payoff function, the semi–flow induced by the BNN dynamics is weakly continuous. In Section 3 we prove that, as in the finite case, the rest points of the dynamics coincide with the Nash equilibria (this property is called Nash stationarity) if the payoff function is continuous and the strategy space is a compact metric space.

The main goal of the paper is to characterize dynamic stability of the BNN dynamics by applying static stability concepts to the equilibria. There turn out to be some important differences between the case of a finite number of pure strategies (as studied by [1] and [20]) and the continuous strategy case studied here. Probably most important is the fact that strict equilibria are not necessarily (Lyapunov) stable in the continuous case.[3] We demonstrate this in Section 4 through an example with a quadratic payoff function. Interestingly, static stability concepts originally developed for the replicator dynamics (like a "continuously stable strategy" (CSS) [13] and Evolutionary Robustness [25]) become relevant for stability with respect to the BNN dynamics. These concepts are introduced in Section 5. In Section 6 and 7 we deal with two classes of games, namely doubly symmetric games and negative semi–definite games, for which we have nearly complete results with respect to stability. Some proofs are collected in an Appendix.

2 The BNN dynamics

We consider symmetric two–player games with (pure) strategy set S. Let $\mathcal{A}$ be a σ–algebra on S and μ be a finite measure on $(S, \mathcal{A})$. Let $f : S \times S \to \mathbb{R}$ be a bounded measurable function, where $f(x, y)$ is the payoff for player 1 when he plays x and player 2 plays y. An interesting special case which will be treated in more detail is when S is a compact interval $S \subset \mathbb{R}$ with the Lebesgue measure.

A *population* is identified with the aggregate play of its members and is described by a probability measure P on the measurable space $(S, \mathcal{A})$. We denote by Δ the set of all populations (probability measures or mixed strategies) on S. Since Δ is not a vector space, we shall work with the linear span of Δ, that is the space $\mathcal{M}^e(S, \mathcal{A})$ of all finite and signed measures. Recall that ν is a finite signed measure on $(S, \mathcal{A})$ if there are two finite measures μ^1 and μ^2 such that for all sets $A \in \mathcal{A}$, $\nu(A) = \mu^1(A) - \mu^2(A)$.

The average payoff of a measure P against a measure Q is defined as

$$E(P, Q) = \int_S \int_S f(x, y) Q(dy) P(dx) \,. \tag{3}$$

Let

$$\sigma(x, P) := E(\delta_x, P) - E(P, P)$$

[3] This is also the case for other dynamics like the replicator dynamics, see e.g. Oechssler and Riedel, [24] and [25].

denote the difference between the payoff of strategy $x \in S$ (identified with the Dirac measure δ_x on x) and the average population payoff. The *excess payoff* of pure strategy x when matched against population P is defined as

$$\sigma_+(x, P) := \max(\sigma(x, P), 0).$$

We now define the *Brown–von Neumann–Nash dynamics* on the measure space $(S, \mathcal{A}, \mu)$ as the differential equation on Δ

$$\dot{P}(A) = \int_A \sigma_+(x, P)\mu(dx) - P(A) \int_S \sigma_+(x, P)\mu(dx), \tag{4}$$

for all $A \in \mathcal{A}$. Let $\Sigma(P) := \int_S \sigma_+(s, P)\mu(ds)$ denote the *total excess.* If $\Sigma(P) > 0$, then the *relative excess* for a subset $A \in \mathcal{A}$ is denoted by $R^P(A) := \frac{1}{\Sigma(P)} \int_A \sigma_+(x, P)\mu(dx)$ and defines a probability measure on $(S, \mathcal{A})$, absolutely continuous with respect to μ, with density function $r^P(x) = \frac{1}{\Sigma(P)}\sigma_+(x, P)$. Then (4) can be rewritten as

$$\dot{P}(A) = \Sigma(P)(R^P(A) - P(A)). \tag{5}$$

Hence, under the BNN dynamics, a population P moves toward its relative excess measure R^P, and the speed of motion is proportional to the total excess. For later reference note that by construction of R^P we have that

$$E(R^P, P) \geq E(P, P), \forall P. \tag{6}$$

Since results on existence and uniqueness of solutions are usually not stated for differential equations on measure spaces, we address this issue first.

Theorem 1 *For each $P = P(0) \in \Delta$ there is a unique solution $P(t) \in \Delta$ of the ordinary differential equation (4) for $t \in [0, \infty[$.*

Proof. see Appendix.

Given that a unique solution to the BNN dynamics exists, we can define the semiflow

$$B : \Delta \times [0, \infty[\to \Delta,$$

where $B(P,t) = P(t)$ denotes the population at time t when the BNN dynamics start in $P = P(0)$.

In most applications, S is a metric space and then the weak topology on Δ is a natural choice. For several reasons it is useful to know whether the semiflow B is weakly continuous. First, from an applied perspective, a starting point $P(0)$ can only be known as a rough approximation. Thus, it would be reassuring to know that dynamics that start at nearby initial points, do not diverge from each other too much. Second, a continuous model as we use it here is only employed for convenience. A continuous model should always be a good approximation for a finite model if the number of strategies gets large. Weak continuity of the flow is a sufficient condition for such an approximation to persist over time (compare [25]). Finally, from a mathematical perspective, for S a compact metric space, B is a continuous semiflow on a compact metric space Δ, for which we can then employ a large body of dynamical systems theory, in particular make use of ω-limits to describe the asymptotic behavior.

Theorem 2 *Let S be a separable metric space and f be bounded and Lipschitz continuous. Then the semiflow B is continuous with respect to the weak topology of measures.*

Proof. see Appendix.

3 Nash stationarity

A nice property of the BNN dynamics is that for continuous f (and thus, in particular, for the finite strategy case) the rest points of the dynamics coincide with the Nash equilibria.[4] The total excess $\Sigma(P)$ vanishes if $\sigma(x,P) \leq 0$ or $E(\delta_x, P) \leq E(P,P)$ for μ–almost all $x \in S$, in particular, if P is a Nash equilibrium. For continuous payoff functions f the reverse holds also.

[4] Sandholm [27] calls this property “Nash stationarity”.

Proposition 1 *Let S be a compact metric space, μ a finite Borel measure on S with full support. Suppose f is continuous. Then P is a rest point of the BNN dynamics if and only if (P, P) is a Nash equilibrium.*

Proof. If P is a best reply to itself, then $\sigma_+(x, P) = 0$ for all x, and stationarity follows.

Let P^* be a stationary point of (8), that is

$$\int_A \sigma_+(x, P^*)\mu(dx) = P^*(A)\Sigma(P^*) \tag{7}$$

for all Borel sets A. We distinguish two cases, $\Sigma(P^*) = 0$ and $\Sigma(P^*) > 0$.

Case 1: $\Sigma(P^*) = 0$. In this case, for $\mu-$ almost every x, we have

$$\sigma_+(x, P^*) = 0.$$

$\sigma_+(x, P^*)$ inherits continuity from f. As μ has full support, it follows that $\sigma_+(x, P^*) = 0$ holds true for all $x \in S$. This is equivalent to

$$E(\delta_x, P^*) \leq E(P^*, P^*),$$

and it follows that P^* is a best reply to itself.

Case 2: $\Sigma(P^*) > 0$. Since P^* is a stationary point of (8), we get from (7) that P^* has a density p^* with respect to Lebesgue measure and

$$p^*(x) = \frac{\sigma_+(x, P^*)}{\Sigma(P^*)}$$

for P^*-almost every x. For every x with $p^*(x) > 0$, we have thus

$$\sigma_+(x, P^*) > 0,$$

or

$$E(\delta_x, P^*) > E(P^*, P^*).$$

By integrating, we get

$$E(P^*, P^*) = \int_{\{x:p^*(x)>0\}} E(\delta_x, P^*)p^*(x)\mu(dx) > E(P^*, P^*),$$

a contradiction. Hence, we cannot have $\Sigma(P^*) > 0$ for a stationary point P^*. This concludes the proof. ■

4 An example: quadratic games

In the previous section we saw that all symmetric Nash equilibria are rest points of the BNN dynamics. However, some of those Nash equilibria may turn out to be unstable. One is used to think of strict Nash equilibria as particularly stable with respect to all kinds of dynamics. And indeed, in the case of finite strategy sets S it is straightforward to show that strict Nash equilibria are asymptotically stable with respect to the BNN dynamics (see e.g. Berger and Hofbauer [1]). The following simple example shows that this is not the case anymore for general S.

For this example we shall assume that S is a compact interval in $\mathbb{R}$ endowed with the Lebesgue measure. Thus, (4) can be written as

$$\dot{P}(A) = \int_A \sigma_+(x, P)dx - P(A) \int_S \sigma_+(x, P)dx. \tag{8}$$

Example 1 *Let $S \subset \mathbb{R}$ be an interval around 0 and $f(x, y) = -x^2 + axy$ be a linear–quadratic game with $a > 0$. For all parameters a, $(0, 0)$ is a strict Nash equilibrium. However, for $a > 2$, this strict Nash equilibrium is unstable with respect to the BNN dynamics. For $a < 2$, BNN dynamics globally converge to the strict Nash equilibrium as both mean and variance converge to 0 along any solution of BNN.*

Proof. Note first that the game with payoff function $f(x, y) = -x^2 + axy$ is strategically equivalent to the doubly symmetric game with payoff function $f(x, y) = -x^2 + axy - y^2 = f(y, x)$. The behavior of BNN is the same under both payoff functions since $\sigma_+(x, P)$ is the same for both payoff functions. Let $P_i := \int_S x^i P(dx)$ denote the ith moment of P. Then

$$E(\delta_x, P) = \int_S (-x^2 + axy - y^2)P(dy) = -x^2 + axP_1 - P_2 \qquad (9)$$

and

$$\begin{aligned} E(P, P) &= \int_S \int_S (-x^2 + axy - y^2)P(dx)P(dy) \\ &= aP_1^2 - 2P_2. \end{aligned} \qquad (10)$$

Therefore

$$\sigma_+(x, P) = [x(aP_1 - x) - aP_1^2 + P_2]_+ \qquad (11)$$

and the density $r^P(x, P)$ of the relative excess measure is the positive part of a quadratic function[5] in x. Both the maximizer and the mean are at $x = \frac{a}{2}P_1$. Hence the mean value of P changes under (8) according to

$$\dot{P}_1 = \Sigma(P)\left(\frac{a}{2} - 1\right)P_1. \qquad (12)$$

Thus, for $a < 2$, $P_1(t) \to 0$, whereas for $a = 2$, $P_1(t) \equiv P_1(0)$, and for $a > 2$, $P_1(t)$ moves away from 0. Hence, for $a > 2$, δ_0 is unstable with respect to the BNN dynamics.

To prove asymptotic stability for $a < 2$, note that (10) implies

$$P_2 = \frac{1}{2}aP_1^2 - \frac{1}{2}E(P, P).$$

Thus,

$$\dot{P}_2 = aP_1\dot{P}_1 - \frac{1}{2}\frac{d}{dt}E(P, P).$$

Since the game is doubly symmetric, we have furthermore that

$$\begin{aligned} \frac{d}{dt}E(P(t), P(t)) &= E(\dot{P}, P) + E(P, \dot{P}) = 2E(\dot{P}, P) \\ &= 2\Sigma(P)E(R^P - P, P) \geq 0, \end{aligned} \qquad (13)$$

[5] We ignore here boundary effects, assuming essentially $S = \mathbb{R}$. For a compact interval S the result follows from the analysis in section 6, see Example 4.

where the last inequality follows from (6). By (12) we obtain

$$\begin{aligned}\dot{P}_2 &= aP_1\Sigma(P)\left(\frac{a}{2}-1\right)P_1 - 2\Sigma(P)E(R^p - P, P) \\ &= \Sigma(P)\left[aP_1^2\left(\frac{a}{2}-1\right) - 2E(R^p - P, P)\right] \le 0.\end{aligned}$$

The second term in the bracket has the required sign by (13). The first term is negative for $0 < a < 2$. The inequality is strict unless $P = \delta_0$, which proves that P_2 is a Lyapunov function for the BNN dynamics. ■

The fact that the parameter a is decisive for stability, suggests that second derivatives of f may play an important role. The following sections show that this intuition is correct.

The above example also shows that δ_0 is unstable in the strong topology. Even if the initial measure has some positive mass on 0, this will disperse into a smooth distribution of better replies near 0 and the mass at 0 will decrease to 0, see Figure 1 for a numerical example. This is in contrast to the replicator dynamics. In the above example 0 is an “uninvadable” strategy (in the sense of Bomze [6]), which implies that it is stable in the strong topology with respect to the replicator dynamics for every $a \in \mathbb{R}$, see [24, Theorem 3].

5 Stability and the measure of closeness

Most, if not all, relevant strategy spaces carry an appropriate metric. For subsets of $\mathbb{R}^n$, there is the Euclidean distance. When considering Bayesian games, strategies are given by certain classes of functions that also come with metrics. For this reason, we assume from now on that S carries a metric d.

The choice of topology is an important issue when defining dynamic stability as one has to specify what it means for a populations Q to be “close” to a given population P. See Oechssler and Riedel [25] for an extensive discussion on this. For the reasons stated there, we find it most appropriate to use the topology of weak convergence to measure closeness of populations in evolution. Accordingly, P_n converges weakly to P if $\int_S f dP_n \to \int_S f dP$

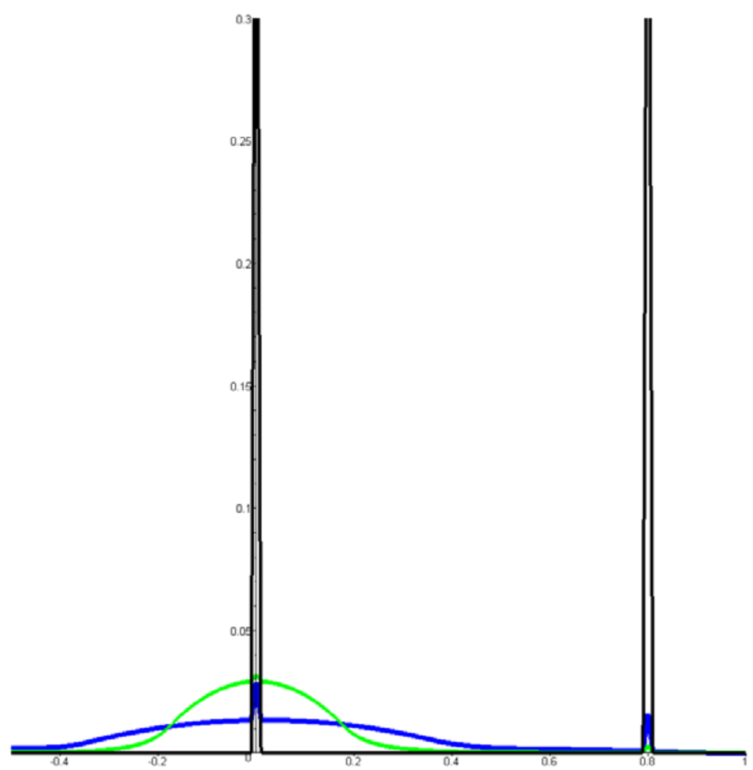

Figure 1: Simulated BNN dynamics for the payoff function $f(x, y) = -x^2 - 2xy$. The initial population has mass 0.5 on the two points 0 and .8. After a few steps of the discretized BNN dynamics, the point masses have decreased drastically and the distribution is dispersed between -0.4 and .4. The grey curve shows that after 100 steps the distribution starts concentrating around the long run equilibrium 0. In fact, we have convergence with respect to the weak topology, but no convergence with respect to the strong topology.

for every bounded, continuous real function f. The Prohorov metric can be used to measure the distance between populations. It is defined as (cf. [4, p. 238])

$$\rho(P,Q) := \inf\left\{\varepsilon > 0 : Q(A) \leq P(A^\varepsilon) + \varepsilon \text{ and } P(A) \leq Q(A^\varepsilon) + \varepsilon,\, \forall A \in \mathcal{A}\right\},$$

where $A^\varepsilon := \{x : \exists y \in A,\, d(y,x) < \varepsilon\}$. Thus, P_n converges weakly to P if and only if $\rho(P_n, P) \to 0$.

The weak topology captures the following notion of closeness. If $Q = (1-\varepsilon)\delta_u + \varepsilon\delta_x$ with $0 \leq \varepsilon \leq 1$, then $\rho(\delta_u, Q) = \min\{\varepsilon, d(u,x)\}$. Thus, population Q is close to the Dirac measure δ_u only if a small subpopulation deviates to a (possibly far away) pure strategy x or if a (possibly large) part of the population deviates to a nearby strategy x. In particular, the distance between two homogenous populations agrees with the natural metric on the set of pure strategies, i.e. $\rho(\delta_u, \delta_x) = d(u,x)$, when u and x are close to each other.

The next definition specifies the *dynamic* stability concepts we will use in the following.

Definition 1 *Let $Q^\star$ be a rest point of the BNN dynamics. Then*

- $Q^\star$ *is called* (Lyapunov) stable *if for all $\varepsilon > 0$ there exists an $\eta > 0$ such that $\rho(Q(0), Q^\star) < \eta \Rightarrow \rho(Q(t), Q^\star) < \varepsilon$ for all $t > 0$.*
- $Q^\star$ *is called* asymptotically stable *if additionally there exists $\varepsilon > 0$ such that $\rho(Q(0), Q^\star) < \varepsilon \Rightarrow \rho(Q(t), Q^\star) \to 0$.*

Dynamic stability can be related to a number of static stability concepts which have the advantage that they can easily be checked given the payoff function. Since strictness of Nash equilibrium is not sufficient for dynamic stability, stronger concepts are required. As it turns out, concepts originally developed for the continuous version of the replicator dynamics in evolutionary biology like CSS [13] and Evolutionary Robustness [25] become relevant for the BNN dynamics as well.

The classical definition of an evolutionary stable strategy (ESS) (Maynard Smith [21]) requires that for all mutant populations R there exists an invasion barrier ε such that the original population P does better against the mixed population $(1-\eta)P+\eta R$ than R does for all $\eta \leq \varepsilon$. In this definition some invasion barrier exists for each R.

Eshel and Motro [13] introduced the following definition for $S \subset \mathbb{R}$.

Definition 2 (CSS) *A strategy u is a* continuously stable strategy (CSS) *if (1) it is an ESS and (2) there exists an $\varepsilon > 0$ such that for all v with $|v-u| < \varepsilon$ there exists an $\eta > 0$ such that for all x with $|v-x| < \eta$*

$$f(v,x) > f(x,x) \text{ if and only if } |v-u| < |x-u|.$$

As shown by Eshel [14] if f is twice differentiable, a necessary condition for an ESS u to be a CSS is that

$$f_{xx}(u,u) + f_{xy}(u,u) \leq 0. \tag{14}$$

Condition (14) is sufficient if the weak inequality is replaced by a strict one.

The following condition was introduced by Oechssler and Riedel [25] and is stronger than CSS.

Definition 3 *A population $P^* \in \Delta(S)$ is* evolutionarily robust *if there exists $\varepsilon > 0$ such that for all $Q \neq P^*$ with $\rho(Q,P^*) < \varepsilon$ we have*

$$E(P^*,Q) > E(Q,Q). \tag{15}$$

When (15) holds for all $Q \neq P$, P is called *globally evolutionarily robust.*

6 Doubly symmetric games

Games in which all players have the same payoff function f and which have a symmetric payoff function, $f(x,y) = f(y,x)$ for all $x, y \in S$, are called doubly symmetric. Doubly symmetric games or games that can be transformed into the symmetric form (as the one in Example 1) have the property that the mean payoff $E(P,P)$ is increasing along every solution of BNN.

Lemma 1 *Let $(S, \mathcal{A}, \mu)$ be a measure space with μ a finite measure on S. Consider a doubly symmetric game. Then the mean payoff $E(P, P)$ is monotonically increasing along every solution of BNN, and strictly increasing along every nonstationary solution. A local maximizer of mean payoff is stable under BNN. If (S, d) is a compact metric space, μ a finite Borel measure on S with full support and f is Lischitz continuous, then the set of limit points of any trajectory is a nonempty connected compact set of Nash equilibria.*

Proof. The fact that mean payoff $E(P, P)$ is monotonically increasing along every solution of BNN follows directly from (13). To prove (asymptotic) stability of (strict) local maxima of mean payoff, we show that $\Lambda(Q) := E(P^*, P^*) - E(Q, Q)$ is a Lyapunov function. By (13) we have that

$$\dot{\Lambda}(Q) = -\frac{d}{dt}E(P(t), P(t)) \leq 0.$$

The result then follows from a suitable generalization of Lyapunov's theorem (see e.g. Oechssler and Riedel [25, Appendix B] or Bhatia and Szegö [3]).

If S is compact metric space, then the set of limit points of any trajectory is non–empty. If additionally f is Lipschitz continuous, Theorem 2 implies that the semiflow is weakly continuous and from standard results in dynamic systems theory (see e.g. [3]) the set of limit points is compact and connected. By the above Lyapunov function each ω–limit point of a trajectory is stationary and hence by Proposition 1 a Nash equilibrium. ■

We use the above general result to show local or global asymptotic stability of an equilibrium. We will demonstrate this for two classes of games: Games with an equilibrium that satisfies evolutionary robustness and games with a unique Nash equilibrium.

Proposition 2 *Consider a doubly symmetric game with Lipschitz continuous payoff function f and compact metric strategy space S. If P^* is evolutionarily robust, then P^* is asymptotically stable with respect to BNN.*

Proof. By definition of evolutionary robustness, we have for Q close to P^*,

$$\begin{aligned}\Lambda(Q) &= E(P^*, P^*) - E(Q, Q) \\ &= E(P^*, P^*) - E(P^*, Q) + E(P^*, Q) - E(Q, Q) \\ &\geq E(P^*, P^*) - E(P^*, Q) \\ &= E(P^*, P^*) - E(Q, P^*) \geq 0,\end{aligned}$$

where the last equality follow from double symmetry of f, and the last inequality from the fact that every evolutionary robust population is a symmetric Nash equilibrium. Note that the first inequality above becomes strict unless $Q = P^*$. The result then follows from Lemma 1. ■

For replicator dynamics and finite (double symmetric) games, ESS is equivalent to asymptotic stability. So one might conjecture that evolutionary robustness is equivalent to asymptotic stability for BNN here. But we know already that this is not true for BNN dynamics. In fact, for quadratic games, CSS (which is weaker than evolutionary robustness) is necessary and sufficient for asymptotic stability. Interestingly, one can show that CSS is always necessary .

Proposition 3 *Let S be an interval in $\mathbb{R}$, with x^* in the interior of S, and let f be twice continuously differentiable and symmetric. If δ_{x^*} is asymptotically stable with respect to BNN, then x^* satisfies the CSS condition (14).*

Proof. By (13), every asymptotically stable state δ_{x^*} must correspond to a local maximum of mean payoff $E(P, P)$. In particular, x^* must be a maximum of $f(x, x)$. The necessary second order condition for x^* to be a maximum is $f_{xx}(x^*, x^*) + 2f_{xy}(x^*, x^*) + f_{yy}(x^*, x^*) \leq 0$, which reduces due to symmetry of f to the CSS condition (14). ■

Given the insight from Example 1 and the previous proposition one might hope that for general payoff functions, CSS is sufficient for a homogeneous population $P^* = \delta_{u^*}$ to be asymptotically stable with respect to BNN. Since this result holds for quadratic payoff function, one may further conjecture that the general result can be proven by using a second order Taylor approximation of the payoff function. The following example shows that this is unfortunately not the case in general.

Example 2 *Let $S = [-1, 1]$ and $f(x, y) = 10x^4 - x^2 - xy$.*

Let $g(x, y) = -x^2 - xy$ be the quadratic approximation (through a second order Taylor approximation) at $x = y = 0$. Since $g(x, y)$ satisfies condition (16), Theorem 3 below implies that δ_0 is globally asymptotically stable for the payoff function $g(x, y)$. However, for the actual payoff function f, the BNN dynamics converges to $Q = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ from any initial $P(0) \neq \delta_0$. This follows from Proposition 4 or Theorem 3 since Q is the unique Nash equilibrium of this negative semi–definite game. Figure 2 shows a typical simulation of time–discretized BNN dynamics where the initial population is a discretized, truncated normal distribution whose mean can be arbitrarily close to 0. Clearly, the dynamics diverges to $Q = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$.

Note that $(0, 0)$ is not a Nash equilibrium for the payoff function $f(x, y)$, and hence not even a stationary point under BNN. However it is a local strict Nash equilibrium and satisfies condition (14) for CSS. Hence, δ_0 is stable for the replicator dynamics w.r.t. initial distributions with support close to 0 and attracts such initials whose support is an interval containing 0, see [9, 10].

Proposition 4 *Let S be compact and f Lipschitz continuous. If a doubly symmetric game has a unique Nash equilibrium P^*, then P^* is globally asymptotically stable under BNN.*

Proof. Follows directly from Lemma 1. ■

The class of doubly symmetric games may appear restrictive (and it is) but there are games that can easily be transformed into a such game as the following example shows.

Example 3 *Consider a Bertrand game with heterogeneous goods. Two firms simultaneously set prices p_1 and p_2. The demand function for firm*

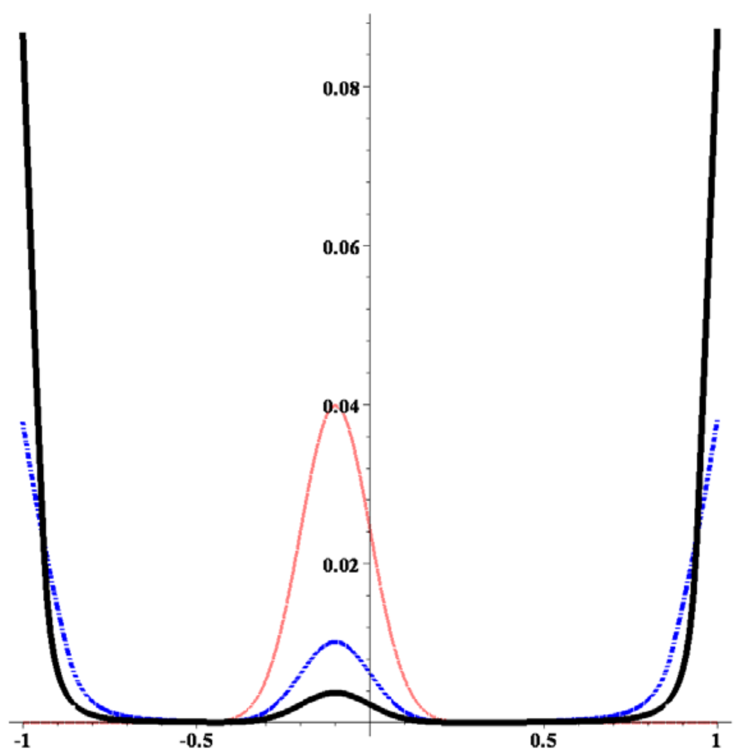

Figure 2: Simulation of time–discretized BNN dynamics with payoff function $f(x, y) = 10x^4 - x^2 - xy$ on strategy space $\{-1, -.99, \ldots, .99, 1\}$. Initial distribution (light grey) is truncated normal. Dashed line shows simulation after 10 iterations, solid black line after 50 iterations.

i given by $1 - p_i + \gamma p_{-i}$ *with* $0 < \gamma < 2$. *Firm 1's payoff function is then*

$$f(p_1, p_2) = p_1(1 - p_1 + \gamma p_2).$$

By adding $p_2 - p_2^2$ *to the payoff function, this game can be transformed into a doubly symmetric one. Thus, by Proposition 4 BNN dynamics globally converge (also for the original game) to the unique symmetric Nash equilibrium* $p_1^* = p_2^* = \frac{1}{2-\gamma}$.

Generalizing the preceding example, we can now completely characterize the dynamic behavior of BNN dynamics for quadratic games.

Example 4 *Let* $S = [-A, B] \subset \mathbb{R}$ *be an interval around* 0 *and* $f(x, y) = -x^2 + axy - y^2$ *be a linear–quadratic game. For all parameters* $a \in \mathbb{R}$, $(0, 0)$ *is a strict Nash equilibrium. For* $a < 2$, 0 *is a strict maximizer of mean payoff and BNN dynamics globally converge to* δ_0. *For* $a > 2$, *there are two other symmetric strict Nash equilibria at the boundary of S:* $(-A, -A)$ *and* (B, B). δ_{-A} *and* δ_B *are local maximizers of mean payoff and hence asymptotically stable under BNN. In the case* $a = 2$, *there is a continuum of pure strategy Nash equilibria* (x, x) *for all* $x \in S$ *and BNN dynamics converge to this set of Nash equilibria.*

7 Negative semi–definite games

In this section we will consider games with an expected payoff function that is negative semi–definite in the sense that for all $P, Q \in \Delta$

$$E(P - Q, P - Q) \leq 0 \tag{16}$$

(see e.g. [20] for the corresponding property in finite games; Sandholm [27] calls this class 'stable games'). Note first that linear quadratic games like $f(x, y) = -x^2 + axy$ satisfy condition (16) if and only if $a \leq 0$ as one can easily check. Furthermore, every symmetric zero–sum game satisfies condition (16). By definition of a symmetric zero–sum game, $f(x, y) +$

$f(y,x) = 0$ for all $x, y \in S$. This implies that $E(P,Q)+E(Q,P) = 0$, and in particular $E(P,P) = 0$. Therefore, $E(P-Q, P-Q) = 0$. Further examples for negative semi–definite games include contests (see Example 5 below) and the War of Attrition (see Example 6). Finally, it is well known (see e.g. [19, p. 122]) that all finite games with an interior ESS satisfy (16) with strict inequality.

Lemma 2 *(1) Under condition (16), the set of Nash equilibria is convex.*

(2) If either there exists a strict Nash equilibrium or condition (16) holds with strict inequality (and at least one NE exists), there is a unique Nash equilibrium, which is, furthermore, globally evolutionarily robust.

Proof. (1) Suppose P^* and Q^* are Nash equilibria. By condition (16) we have that

$$E(P^*,P^*)+E(Q^*,Q^*) \leq E(P^*,Q^*)+E(Q^*,P^*) \leq E(Q^*,Q^*)+E(Q^*,P^*),$$

which implies that $E(P^*,P^*) = E(Q^*,P^*)$. Thus, any convex combination $P_\lambda = \lambda P^* + (1-\lambda)Q^*$ is also a best reply against P^*, and similarly against Q^*. Since for all Q

$$\begin{aligned} E(P_\lambda, P_\lambda) &= \lambda E(P_\lambda, P^*) + (1-\lambda) E(P_\lambda, Q^*) \\ &\geq \lambda E(Q, P^*) + (1-\lambda) E(Q, Q^*) \\ &= E(Q, P_\lambda), \end{aligned}$$

P_λ is also a Nash equilibrium which proves that the set of Nash equilibria is convex.

(2) Let P^* be a Nash equilibrium and $Q \neq P^*$. Then

$$E\left(P^*-Q, Q\right) = E\left(P^*-Q, Q-P^*\right) + E\left(P^*-Q, P^*\right).$$

The first term is nonnegative by condition (16) and the second term is nonnegative by definition of a Nash equilibrium. For a strict Nash equilibrium, the second term is strictly positive. If (16) holds with strict inequality, the first term is strictly positive. In either case, $E\left(P^*-Q, Q\right) > 0$, that is, P^*

is globally evolutionarily robust. This in turn implies that there is no other Nash equilibrium because $E(Q,Q) < E(P^*,Q)$ for all $Q \neq P^*$. ■

We may now proceed to study the global stability properties of Nash equilibria in negative semi–definite games.

Theorem 3 *For negative semi–definite games (16), define the function*

$$H(P) = \frac{1}{2}\int_S \sigma_+(x,P)^2\, \mu(dx).$$

The following statements hold true:
(1) H is nonnegative and decreases to 0 *along every solution of BNN.*
(2) If S is a compact metric space, f is continuous, and μ a measure with full support, then every trajectory of BNN converges to the set of Nash equilibria.
(3) In particular, every strict Nash equilibrium and every equilibrium that satisfies (16) with strict inequality is globally asymptotically stable.

Proof. Let us first determine the gradient of $\sigma(x,P)$ with respect to P at some point Q. We have

$$\nabla\sigma(x,P)(Q) = E(\delta_x,Q) - E(P,Q) - E(Q,P).$$

From this, we obtain via the chain rule

$$\begin{aligned}
\frac{d}{dt}H(P) &= \int_S \sigma_+(x,P)\,\nabla\sigma(x,P)(\dot{P})\mu(dx) \\
&= \int_S \sigma_+(x,P)\left[E\left(\delta_x,\dot{P}\right) - E\left(P,\dot{P}\right) - E\left(\dot{P},P\right)\right]\mu(dx) \\
&= \Sigma(P)\left(E\left(R^P,\dot{P}\right) - E\left(P,\dot{P}\right) - E\left(\dot{P},P\right)\right),
\end{aligned}$$

where we have used the definition of the relative excess measure R^P. By definition of the dynamics $\dot{P}_t$, we proceed to

$$\begin{aligned}
\frac{d}{dt}H(P) &= \Sigma(P)^2\left(E\left(R^P,R^P-P\right) - E\left(P,R^P-P\right) - E\left(R^P-P,P\right)\right) \\
&= \Sigma(P)^2\left(E\left(R^P-P,R^P-P\right) - E\left(R^P-P,P\right)\right).
\end{aligned}$$

The first term in parentheses, $E\left(R^P - P, R^P - P\right) \leq 0$ by Assumption (16), and the second term, $E\left(R^P - P, P\right) \geq 0$ by definition of the relative excess measure (see (6)). Thus, we obtain

$$\frac{d}{dt}H(P) \leq 0$$

and the inequality is strict whenever P is not a stationary point.

If S is a compact metric space and f Lipschitz continuous, then Δ is compact (in the weak topology) and BNN generates a weakly continuous semiflow by Theorem 2. Hence, the ω–limit set (in the weak topology) of a trajectory $P(t)$ is non-empty and contained in the set of $P \in \Delta$ for which $\frac{d}{dt}H(P) = 0$, which is the set of Nash equilibria by Proposition 1. Thus, the BNN dynamics converge to the convex set of Nash equilibria. In particular, every strict Nash equilibrium and every equilibrium that satisfies (16) with strict inequality, is a unique equilibrium and, therefore, globally asymptotically stable.

For general measure spaces S, we proceed similar to [7]. The above expression implies

$$\begin{aligned}
\frac{d}{dt}H(P) &\leq -\Sigma\left(P\right)^2 E\left(R^P - P, P\right) \\
&= -\Sigma\left(P\right)^2 \left(\int E(\delta_x, P)\frac{\sigma_+(x,P)}{\Sigma\left(P\right)}\mu(dx) - E(P,P)\right) \\
&= -\Sigma\left(P\right)^2 \int \left[E(\delta_x, P) - E(P,P)\right]\frac{\sigma_+(x,P)}{\Sigma\left(P\right)}\mu(dx) \\
&= -\Sigma\left(P\right)^2 \int \frac{\sigma_+(x,P)^2}{\Sigma\left(P\right)}\mu(dx) = -2\Sigma(P)H(P). \qquad (17)
\end{aligned}$$

Since f is bounded, $\sigma_+(x,P) \leq 2\left\|f\right\| =: \frac{1}{c}$. Hence,

$$\Sigma(P) = \int_S \sigma_+(x,P)\mu(dx) \geq c\int_S \sigma_+(x,P)^2\mu(dx) = cH(P).$$

Therefore, (17) implies the differential inequality

$$\frac{d}{dt}H(P) \leq -2cH(P)^2,$$

which integrates to $H(P(t)) \leq \frac{H(P(0))}{1+2cH(P(0))t}$. Hence $H(P(t)) \to 0$ as claimed.
■

Example 5 *Contests. Let* $S = [a, b]$ *for some numbers* $a < b$*. Two players exert an effort level of* x *and* y *in* S*, respectively, to obtain a prize worth* $K > 0$*. The probability that player 1 wins the prize is* $p(x, y)$*, and the probability that player 2 wins is* $p(y, x) = 1 - p(x, y)$*. Costs are given by some cost function* $c(x)$*. The payoff function is thus* $f(x, y) = Kp(x, y) - c(x)$*. We claim that contests are negative semi–definite. To see this, note that because of* $p(x, y) + p(y, x) = 1$*, we have*

$$E(P, Q) + E(Q, P) = 1 - \int c(x)P(dx) - \int c(y)Q(dy).$$

It follows that $E(P, P) = \frac{1}{2} - \int c(x)P(dx)$*, and this implies*

$$E(P - Q, P - Q) = E(P, P) + E(Q, Q) - E(P, Q) - E(Q, P) = 0.$$

Now assume that $p(\cdot, \cdot)$ *and* $c(\cdot)$ *are continuously differentiable,* $p(\cdot, \cdot)$ *is strictly concave in* x*, and* $c(\cdot)$ *is convex. If there exists* $\bar{x} \in (a, b)$ *with* $K \frac{\partial}{\partial x} p(\bar{x}, \bar{x}) = c'(\bar{x})$*, then* $(\bar{x}, \bar{x})$ *is a strict Nash equilibrium (which is then unique by Lemma 2). By Theorem 3,* $\delta_{\bar{x}}$ *is globally asymptotically stable under BNN dynamics.*

Finally, we will demonstrate how our techniques are useful even when applied to games with discontinuous payoff function, like the war of attrition.

Example 6 *War of attrition. Consider two players fighting for a prize worth* V *to both players. A strategy is to choose a length of time* $x \in [0, M]$ *for which one is prepared to stay in the race. Fighting is costly. The payoffs are given as follows*

$$f(x, y) = \begin{cases} V - y & \text{if } x > y \\ \frac{V}{2} - x & \text{if } x = y \\ -x & \text{if } x < y \end{cases}$$

that is, a player gets the prize if he stays longer in the race than his rival but has to share if they stay equally long.

We assume that $M > V/2$. Otherwise waiting until the end is always profitable. Bishop and Cannings [5] show that there is a unique Nash equilibrium, which has the following equilibrium distribution $P^\star$ with $t^\star = M - V/2$,

$$P^\star([0,x]) = \begin{cases} 1 - e^{-x/V} & \text{if } x \le t^\star \\ 1 - e^{-t^\star/V} & \text{if } t^\star < x < M \\ 1 & \text{if } x = M. \end{cases}$$

Bishop and Cannings [5] show that $P^\star$ is an ESS. They also show [5, p. 118] that

$$E(P-Q, P-Q) = -\int_0^M \left(P\left([s,M]\right) - Q\left([s,M]\right)\right)^2 ds. \tag{18}$$

In particular, the war of attrition is a negative semi–definite game. As the payoff function f is not continuous, we cannot apply the second part of the above theorem. Nevertheless, its conclusion still holds true provided the measure μ that defines the excess measure in the definition of BNN dynamics (see equation (4)) puts some weight on the point M. The intuition for this assumption is as follows: the Nash equilibrium has a mass point on M, but strategies close to M are not being played in equilibrium. If μ is the Lebesgue measure, the excess measure has a density with respect to the Lebesgue measure that is zero close to M, and strictly positive at M. However, a single value of the density does not contribute to the distribution, and thus, the excess measure puts no weight on M if μ is the Lebesgue measure. Consequently, there is in general no hope that BNN generates some mass on or around M if one uses the Lebesgue measure. Therefore, we assume that μ puts some small mass on M. The following proposition shows that this is sufficient for convergence.

Proposition 5 *Assume that $\mu = dx + \varepsilon\delta_M$ for some (small) $\varepsilon > 0$. In the War of Attrition, every trajectory of BNN converges to the unique Nash equilibrium.*

Proof. Without loss of generality, we set $V = 1$ in the proof. Consider the Lyapunov function $H(P)$ as in Theorem 3, where we take $\mu = dx + \varepsilon\delta_M$, the sum of the Lebesgue measure on $[0, M]$ and a point mass on M. The proof of $H(P(t)) \rightarrow_{t\rightarrow\infty} 0$ does not use continuity of f.

We show next that $H(P)$ is lower semi-continuous in the weak topology in the sense that $H(P) \leq \liminf H(P^n)$ if (P^n) converges in the weak topology to P. By symmetry, we have

$$
\begin{aligned}
1 &= \int\int 1_{\{x<y\}}P(dx)P(dy) + \int\int 1_{\{y<x\}}P(dx)P(dy) + \int\int 1_{\{x=y\}}P(dx)P(dy) \\
&= 2\int\int 1_{\{x<y\}}P(dx)P(dy) + \int\int 1_{\{x=y\}}P(dx)P(dy).
\end{aligned}
$$

It follows that average payoff can be written as

$$
\begin{aligned}
E(P, P) &= \int\int 1_{\{x<y\}}P(dx)P(dy) + \frac{1}{2}\int\int 1_{\{x=y\}}P(dx)P(dy) \\
&\quad - \int\int \min(x, y)P(dx)P(dy) \\
&= \frac{1}{2} - \int\int \min(x, y)P(dx)P(dy).
\end{aligned}
$$

As $\min(x, y)$ is continuous in (x, y), $E(P, P)$ is continuous in the weak topology. For points x with $P(\{x\}) = 0$, $E(\delta_x, P) = P([0, x)) - \int \min(x, y)P(dy)$. By the Portmanteau Theorem, $P \mapsto P([0, x))$ is continuous at P in the weak topology for these x. $\int \min(x, y)P(dy)$ is continuous in the weak topology because the integrand is continuous. We conclude that $\sigma_+(x, P)$ is continuous at P in the weak topology for all x with $P(\{x\}) = 0$. Now let $P^n \longrightarrow P$ in the weak topology. Then $\lim \sigma_+(x, P^n) = \sigma_+(x, P)$ for all points x with $P(\{x\}) = 0$. As the set of points x with $P(\{x\}) = 0$ has full Lebesgue measure, and the payoff function is bounded, we get by dominated convergence that $\lim \frac{1}{2}\int_S \sigma_+(x, P^n)^2\, dx = \frac{1}{2}\int_S \sigma_+(x, P)^2\, dx$. Hence, the first part of H is continuous in the weak topology. Now consider $\sigma_+(M, P) = 1 - \frac{1}{2}P(\{M\}) - \displaystyle\int xP(dx)$. By the Portmanteau Theorem, $P(\{M\}) \geq \limsup P^n(\{M\})$. Therefore, $\sigma_+(M, P) \leq \liminf \sigma_+(M, P^n)$. This finally establishes $H(P) \leq \liminf H(P^n)$.

As H is lower semi-continuous in the weak topology, we conclude that every limit point P^0 of BNN dynamics satisfies $H(P^0) = 0$. It follows that

$\sigma(x, P^0) \leq 0$ for $\mu-$ almost all $x \in [0, M]$. It remains to be shown that this implies $\sigma(x, P^0) \leq 0$ for all x. As μ has a point mass on M, we have $\sigma(M, P^0) \leq 0$. Now consider some $x < M$. There exists a sequence (x^n) that converges to x from the right and satisfies $P^0(\{x^n\}) = 0$ as well as

$$\sigma(x^n, P^0) \leq 0 \text{ for all } n.$$

It follows that

$$\begin{aligned}
E(P^0, P^0) &\geq \lim E(\delta_x^n, P^0) = \lim P^0([0, x^n)) - \int \min(x^n, y) P^0(dy) \\
&= P^0([0, x]) - \int \min(x, y) P^0(dy) \\
&\geq P^0([0, x)) + \frac{1}{2} P^0(\{x\}) - \int \min(x, y) P^0(dy) \\
&= E(\delta_x, P^0).
\end{aligned}$$

This establishes $\sigma\left(x, P^0\right) \leq 0$ for all $x < M$. ■

Appendix

Proof of Theorem 1

The strategy for proving the Theorem is the following. Denote by

$$F(Q) := \int_{\cdot} \sigma_+(x, Q) \mu(dx) - Q(\cdot) \int_S \sigma_+(x, Q) \mu(dx)$$

the right–hand side of the BNN–dynamics. Since F is neither bounded nor globally Lipschitz continuous on $\mathcal{M}^e$, we construct in the following lemma an auxiliary function $\tilde{F}$ which has these properties and coincides with F on Δ (see also Bomze, [6]). In particular, we show that $\tilde{F}$ satisfies a global Lipschitz condition

$$\exists K > 0 \text{ s.t. } \forall \mu, \nu \in \mathcal{M}^e, \left\| \tilde{F}(\mu) - \tilde{F}(\nu) \right\| \leq K \left\| \mu - \nu \right\|,$$

where $\|\cdot\|$ denotes the variational norm on $\mathcal{M}^e(S, \mathcal{A})$. The variational norm is given by

$$\|\mu\| = \sup_g \left| \int g \, d\mu \right|,$$

where the sup is taken over all measurable functions $g : S \to \mathbb{R}$ bounded by 1, $\sup_{s\in S} |g(s)| \le 1$. Endowed with the variational norm, $\mathcal{M}^e$ is a Banach space (see [30]).

Standard arguments (see e.g. Zeidler [35, Corollary 3.9]) then imply that the ordinary differential equation

$$\dot{Q}(t) = \tilde{F}(Q(t)),\ Q(0) = P$$

has a unique solution $(Q(t))$. Finally, since $\dot{Q}(t)(S) = 0$, $Q(t)$ never leaves Δ, which implies that $(Q(t))$ also solves differential equation (4) on Δ.

Lemma 3 *Suppose f is bounded, then there exists a bounded, Lipschitz continuous function $\tilde{F} : \mathcal{M}^e \to \mathcal{M}^e$, which coincides with F on Δ,*

$$\tilde{F}(P) = F(P),\ \forall P \in \Delta.$$

Proof. We define $\tilde{F}$ as

$$\tilde{F}(Q) = (2 - \|Q\|)_+ \ F(Q)\,.$$

$\tilde{F}$ is zero for $\|Q\| \ge 2$. It is bounded and coincides with F on Δ because probability measures have norm 1. It remains to show that $Q \mapsto F(Q)$ is Lipschitz for $\|Q\| \le 2$.

The estimates

$$|E(\delta_x, Q)| \le \|f\|_\infty \|Q\|\,, \quad |E(P, Q)| \le \|f\|_\infty \|P\| \, \|Q\| \tag{19}$$

imply that for each $x \in S$, the functions $Q \mapsto \sigma(x, Q)$ and hence also $Q \mapsto \sigma_+(x, Q)$ are Lipschitz (for $\|Q\| \le 2$) with a Lipschitz constant L independent of x. Then the map $Q \mapsto \hat{F}(Q)$ with $\hat{F}(Q)(A) = \int_A \sigma_+(x, Q)\mu(dx)$ from $\mathcal{M}^e$ into itself is Lipschitz with Lipschitz constant $L\mu(S)$. In particular, also $Q \mapsto \Sigma(Q) : \mathcal{M}^e \to \mathbb{R}$ is Lipschitz. Hence $F(Q)$ is Lipschitz in Q. ■

Proof of Theorem 2

In the following, we will use the metric $\|\cdot\|_{BL}$ on Δ which metrizes the weak topology (cf. [30, p. 352]). Endowed with the BL–norm, $\mathcal{M}^e$ is a Banach space. For a Lipschitz continuous, bounded function $g : S \to \mathbb{R}$, let

$$\|g\|_{BL} := \sup_{x\in S} |g(x)| + \sup_{x\ne y} \frac{|g(x) - g(y)|}{d(x, y)}. \tag{20}$$

Abusing notation slightly, we define the dual norm $\|\cdot\|_{BL}$ on $\mathcal{M}^e(S,\mathcal{A})$ via

$$\|Q\|_{BL} = \sup\left\{\int g dQ\right\}, \tag{21}$$

where the supremum is taken over all bounded, Lipschitz continuous functions g with $\|g\|_{BL} \leq 1$.

We prove below that we have

$$|\sigma(x,P) - \sigma(x,Q)| \leq L\,\|P - Q\|_{BL} \tag{22}$$

for some constant $L > 0$ and all populations P, Q and all strategies x. The same Lipschitz estimate holds true when we pass to the positive part, so we have

$$|\sigma_+(x,P) - \sigma_+(x,Q)| \leq L\,\|P - Q\|_{BL}\,. \tag{23}$$

This implies that the right hand side of (4) is Lipschitz in the norm (21). The claim follows then from Gronwall's Lemma (see e.g. Zeidler [35, Propositions 3.10 and 3.11]), which implies that

$$\|P(t) - Q(t)\|_{BL} \leq e^{Lt}\,\|P(0) - Q(0)\|_{BL} \tag{24}$$

(for a possibly different L) and hence continuous dependence of solutions on initial conditions for finite time.

It remains to prove the claim (22). By boundedness and Lipschitz continuity of f, there exist constants $L_0, L_1 > 0$ such that for all strategies x, y, x', y'

$$|f(x,y)| \leq L_0 \tag{25}$$

$$\left|f(x,y) - f(x',y')\right| \leq L_1 \max\{d(x,x'), d(y,y')\}. \tag{26}$$

For $x = x'$, the latter inequality yields

$$\left|f(x,y) - f(x,y')\right| \leq L_1 d(y,y'). \tag{27}$$

Let R be a population. Define the function

$$g(y) = \int_S f(x,y)R(dx).$$

Since R is a probability measure, (25) carries over to g,

$$|g(y)| \leq L_0,$$

and so does (27),

$$\left|g(y) - g(y')\right| \leq L_1 d(y, y').$$

Hence, g is a bounded and Lipschitz continuous function with $\|g\|_{BL} \leq L_0 + L_1$. We thus obtain

$$\begin{aligned} |E(R, P-Q)| &= \left|\int_S g(y)(P-Q)(dy)\right| \\ &\leq \|g\|_{BL} \|P-Q\|_{BL} \\ &\leq (L_0 + L_1) \|P-Q\|_{BL}. \end{aligned} \tag{28}$$

By a symmetric argument, we also have

$$|E(P-Q, R)| \leq (L_0 + L_1) \|P-Q\|_{BL}. \tag{29}$$

Now, to prove our claim, note that

$$\begin{aligned} |\sigma(x, P) - \sigma(x, Q)| &\leq |E(\delta_x, P-Q)| + |E(Q, Q) - E(P, P)| \\ &\leq |E(\delta_x, P-Q)| + |E(Q-P, Q)| + |E(P, Q-P)| \end{aligned}$$

Applying (28) for $R = \delta_x$ and $R = P$, as well as (29) for $R = Q$, we finally obtain

$$|\sigma(x, P) - \sigma(x, Q)| \leq 3(L_0 + L_1) \|P-Q\|_{BL},$$

and the proof is complete. ■

References

[1] U. Berger and J. Hofbauer, The Nash Dynamics, mimeo: University of Vienna, 2000.

[2] U. Berger and J. Hofbauer, Irrational behavior in the Brown–von Neumann–Nash dynamics. *Games and Economic Behavior* (2005), in press.

[3] N.P. Bhatia and G.P. Szegö, "Stability theory of dynamical systems," Berlin–Heidelberg–New York: Springer (1970).

[4] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, New York, 1968.

[5] D. Bishop and C. Cannings, A Generalized war of attrition. *Journal of Theoretical Biology* **70**, 85-124 (1978)

[6] I. Bomze, Cross Entropy Minimization in Uninvadable States of Complex Populations, *Journal of Mathematical Biology* **30** (1991), 73-87.

[7] G. Brown and J. von Neumann, Solutions of Games by Differential Equations, in: H. Kuhn and A. Tucker (eds.) "Contributions to the Theory of Games", Annals of Mathematical Studies No. 24, Princeton University Press, Princeton, 1950.

[8] R. Cressman, Stability of the replicator equation with continuous strategy space, *Math. Soc. Sci.* **50** (2005), 127-147.

[9] R. Cressman, J. Hofbauer, Measure dynamics on a one-dimensional continuous trait space: Theoretical foundations for adaptive dynamics, *Theoretical Population Biology* **67** (2005), 47-59.

[10] R. Cressman, J. Hofbauer, F. Riedel, Stability of the replicator equation for a single-species with a multi-dimensional continuous trait space, *J. theor. Biology* (2005), in press.

[11] J. Ely and W. Sandholm, Evolution in Bayesian Games I: Theory, in Press, *Games and Economic Behavior.*

[12] J. Ely and O. Yilankaya (2001), Nash Equilibrium and the Evolution of Preferences, *Journal of Economic Theory* 97, 255-272.

[13] I. Eshel and U. Motro, Kin Selection and Strong Evolutionary Stability of Mutual Help, *Theoretical Population Biology* **19** (1981), 420-433.

[14] I. Eshel, Evolutionary and Continuous Stability, *Journal of Theoretical Biology* **103** (1983), 99-111.

[15] W. Güth and M. Yaari (1992), An Evolutionary Approach to Explain Reciprocal Behavior in a Simple Strategic Game, in: *Explaining Process and Change — Approaches to Evolutionary Economics*, ed. U. Witt, Ann Arbor, 23-34.

[16] S. Hart and A. Mas–Colell, A Simple Adaptive Procedure Leading to Correlated Equilibrium, *Econometrica*, **68** (2000), 1127-1150.

[17] S. Hart and A. Mas–Colell, Regret-based continuous-time dynamics, *Games and Economic Behavior* **45** (2003), 375-394.

[18] A. Heifetz, Ch. Shannon, and Y. Spiegel (2002), What to Maximize if You Must, mimeo, University of California, Berkeley.

[19] J. Hofbauer and K. Sigmund, "The Theory of Evolution and Dynamical Systems," Cambridge: Cambridge University Press, 1988.

[20] J. Hofbauer, From Nash and Brown to Maynard–Smith: Equilibria, Dynamics and ESS, *Selection* **1** (2000), 81-88.

[21] J. Maynard–Smith, The Theory of Games and the Evolution of Animal Conflicts, *Journal of Theoretical Biology* **47 (**1974), 209-222.

[22] M. Meertens, J. Potters, H. Reijnierse, A dynamic selection process for normal form games, mimeo: University of Nijmengen, 2003

[23] J. Nash, Non–Cooperative Games, *Annals of Mathematics*, **54** (1951), 286-295.

[24] J. Oechssler and F. Riedel, Evolutionary Dynamics on Infinite Strategy Spaces, *Econ. Theory* **17** (2001), 141-162.

[25] J. Oechssler and F. Riedel, On the Dynamic Foundation of Evolutionary Stability in Continuous Models, *Journal of Economic Theory* **107** (2002), 223-252.

[26] W. Sandholm, Potential Games with Continuous Player Sets, *Journal of Economic Theory*, **97** (2001), 81-108.

[27] W. Sandholm, Stable Games, mimeo: University of Wisconsin, 2005.

[28] W. Sandholm, Excess payoff dynamics and other well-behaved evolutionary dynamics, *Journal of Economic Theory*, In Press.

[29] K. Schlag, Why Imitate, and If So, How? A Boundedly Rational Approach to Multi-armed Bandits, *Journal of Economic Theory,* **78** (1998), 130-56.

[30] A. Shiryaev, "Probability," 2nd ed., New York: Springer Verlag, 1995.

[31] B. Skyrms, "The Dynamics of Rational Deliberation", Harvard University Press, Cambridge, Mass., 1990.

[32] J. Swinkels, Adjustment Dynamics and Rational Play in Games, *Games and Economic Behavior*, **5** (1993), 455-484.

[33] P. Taylor and L. Jonker, Evolutionary Stable Strategies and Game Dynamics, *Mathematical Bioscience* **40** (1978), 145-156.

[34] J. Weibull, The Work of John Nash in Game Theory, *Journal of Economic Theory* **69** (1996), 153-185.

[35] E. Zeidler, "Nonlinear Functional Analysis and its Applications," Vol. I, Springer Verlag, New York, 1986.