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Additively separable welfare functions reconsidered

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Abstract: In this paper I study the optimal control path for a (capital or pollution) stock which accumulates time with a lag to its control. It is shown that the optimal control path is non-monotonic and cyclical in general, but it is monotonic if the objective function is additively separable in the control and the stock variables. On the one hand, this result generalizes the existing time-to-build literature, where time lagged capital accumulation is supposed to be sufficient for cyclical optimal paths. On the other hand, this result challenges the use of additively separable welfare functions as a good approximation to the "real world" in environmental economic problems and raises questions about the design of optimal environmental policies.

Keywords: dynamic economy-environment interaction, general and partial equilibrium analysis, time lagged optimal control, time-to-build

JEL-Classification: Q50, Q53, C61

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1 Introduction

Numerous environmental problems we face today are caused by stocks of pollutants which accumulate with a time lag to their emission. As a prime example, think of chlorofluorocarbons (CFCs), which have been widely used as cooling agents in refrigeration and air conditioning, as propellants in aerosols sprays and foamed plastics, and as solvents for organic matters and compounds. The CFCs have been valued because of their favorable chemical and biological characteristics. They are chemically inert, not inflammable and non-toxic. Unfortunately, in the stratosphere the CFCs cause the depletion of the ozone layer which shields the earth's surface from ultraviolet radiation. Once released, the CFCs need 5–10 years to reach a height of about 30 km, where the depletion of the ozone layer starts. Hence, the stock of stratospheric CFCs reacts to the emissions of CFCs with a lag time of 5–10 years.

In this paper, I analyze in an optimal control framework the system dynamics of an economy with one consumption good, whose production causes a specific pollutant in fixed proportions. Furthermore, these emissions may be abated at increasing marginal costs, and net emissions accumulate with a time lag to a pollution stock, which has a negative impact on society's welfare. The paper, which is (at least to my knowledge) the first to analyze *time lagged* accumulation of *pollution stocks* in an optimal control framework, merges two distinct strands of the economic literature.

First, there is a substantial environmental economic literature on *instantaneous pollution stock* accumulation. While some contributions (e. g. Keeler et al. 1972, Plourde 1972, Forster 1973, Smith 1977, Van der Ploeg and Withagen 1991, and Gradus and Smulders 1993, 1996) analyze steady-state growth in highly aggregated Ramsey-type optimal growth models with environmental pollution, others focus on the complex system dynamics of environmental problems caused by stock pollutants. Falk and Mendelsohn (1993) analyze the optimal emissions of greenhouse gases, Faber et al. (1999: chap. 11) and Winkler (2002) study the structural change of different production sectors which give rise to one stock pollutant, and Aaheim (1999), Michaelis (1999), Moslener and Requate (2001), Jöst et al. (2003) and Baumgärtner et al. (2003) analyze the dynamic interaction of different stock pollutants.

Second, *time lagged capital* accumulation has recently been discussed in the macroeconomic real business cycle theory. Following an idea first posed in Kalecki (1935), Kydland and Prescott (1982) empirically analyze how far time-to-build (i. e. time lagged accumulation of capital) can explain business cycles. While Kydland and Prescott (1982) argue that the time-to-build feature is essential to cyclical fluctuations in their model, this is doubted by Ioannides and Taub (1992). Rustichini (1989) and Asea and Zak (1999) show, in simple models with one capital good (but different lag structure), that the time-to-build feature is the driving force for the cyclical system dynamics.

Although both pollution and capital accumulation problems look quite similar from a mathematical point of view, there is a substantial difference. In the majority of the contributions in environmental economics, the objective function of pollution accumulation problems is considered to be additively separable in the control and the stock variables, whereas in capital accumulation models the control and the stock variables enter the objective function in a more complex way. The additive separability is often justified as an approximation to the "real world", which is supposed to be warranted if the impact of the environmental problem on welfare is small compared to the welfare gains derived from consumption. But if accumulation occurs with a time lag, the functional form of the objective function plays a crucial role for the qualitative characteristics of the optimal path. In fact, I show that cyclical optimal paths can be observed in general, but monotonic paths are optimal if the objective function is additively separable in the control and the stock variables. Furthermore, it is shown that the optimal control for an additively separable approximation to the original – more complex and not additively separable – problem may deviate substantially from the optimal control of the original problem. This result on the one hand challenges the use of additively separable welfare functions as good approximations to the "real world", at least in the case of time lagged stock accumulation, and on the other hand raises questions about the design of optimal environmental policies.

The paper is organized as follows. In section 2 a generic optimal control problem with time lagged stock accumulation is analyzed. In section 3 these results are applied to a time lagged emission control problem, which is solved subject to the original and additively non-separable objective function and to an approximated and additively separable objective function. Section 4 illustrates the results by some numerical examples. Section 5 concludes.

2 Optimal control of time lagged accumulation problems

In this section I analyze an optimal control problem for a generic welfare function W, subject to a time lagged equation of motion. Consider the following maximization problem

$$\begin{aligned}
\max W &= \int_0^\infty F(e(t), s(t)) \exp[-\rho t] \, dt \quad \text{s.t.} \\
\dot{s}(t) &= e(t-\sigma) - \gamma s(t) \ , \\
s(0) &= s_0 \ , \\
e(t) &= \xi(t) \ , \quad t \in [-\sigma, 0) \ , \\
0 &\leq e(t) \ ,
\end{aligned} \tag{1}$$

where ρ denotes the constant, positive discount rate; F is a twice continuously differentiable generic felicity function, which is a function of the control variable e and the stock variable s. Note that the maximization problem (1) can be identified on the one hand with capital accumulation models, in which case e denotes investment and s the capital stock, and on the other hand with environmental economic models, when e denotes the emissions and s the pollution stock. The crucial feature is that the dynamics of the stock s is governed by a *retarded differential-difference* equation; i. e. the variation in the stock s not only depends on parameters evaluated at time t, but also on parameters evaluated at the earlier time t–σ. As a consequence, and in contrast to instantaneous accumulation models, the specification of an initial value s_0 for the stock s is not sufficient for a unique solution. In addition, an initial path ξ for the emissions e in the time interval $[-\sigma, 0)$ has to be specified. Note that the path of the stock in the time interval $t \in [0, \sigma]$ is completely determined by the initial stock s_0, the initial path ξ, and the retarded equation of motion. Thus, time lagged accumulation models exhibit an additional moment of inertia as the stock reacts with a delay to a variation of the control.

2.1 Necessary and sufficient conditions

To solve the resulting optimization problem, I apply the generalized Maximum principle derived in El-Hodiri et al. (1972) for time lagged optimal control problems. One obtains the following present-value Hamiltonian $\mathcal{H}$[1]

$$\mathcal{H}(e(t), s(t)) = F(e(t), s(t)) \exp[-\rho t] + p_s(t+\sigma) e(t) - \gamma p_s(t) s(t) \ , \tag{2}$$

where p_s denotes the costate variable or shadow price of the stock s. Assuming that the Hamiltonian $\mathcal{H}$ is continuously differentiable with respect to e, the following necessary conditions hold for an optimal solution (partial derivatives are indicated by subscripts and only the time argument is stated explicitly):

$$\mathcal{H}_e(t) \; = \; F_e(t) \exp[-\rho t] + p_s(t+\sigma) \; = \; 0 \ , \tag{3}$$
$$\mathcal{H}_s(t) \; = \; F_s(t) \exp[-\rho t] - \gamma p_s(t) \; = \; -\dot{p}_s(t) \ . \tag{4}$$

These necessary conditions are also sufficient for a unique solution if the Hamiltonian $\mathcal{H}$ is strictly concave in e and s, and in addition, the following transversality condition is satisfied:

$$\lim_{t \to \infty} [p_s(t) s(t)] = 0 \ . \tag{5}$$

A sufficient condition for the strict concavity of the Hamiltonian $\mathcal{H}$ is that

$$F_{ii}(t) < 0 \quad \text{and} \quad \det [F_{ij}(t)] > 0 \ , \quad i, j = e, s \ , \tag{6}$$

which will be assumed in the following. The necessary condition (4) is an inhomogeneous linear first-order differential equation which can be unambiguously solved, together with the transversality condition (5), to yield:

$$p_s(t) = \int_t^\infty F_s(t') \exp[-\rho t'] \exp[-\gamma(t' - t)] \, dt' \ . \tag{7}$$

Hence, at the optimum the shadow price of the stock, p_s, equals the aggregated discounted future contributions to the objective function of one additional marginal unit of the stock s. Condition (3) says that at the optimum the marginal cost/benefit of one additional marginal unit of the control e equals the aggregated future benefit/cost of one additional marginal unit of the stock s. As one unit of e accumulates to the stock s delayed by the time lag σ, the shadow price p_s has to be evaluated at time $t+\sigma$.

[1] I do not explicitly control for $e(t) > 0$, which amounts to the assumption that the optimal control path is an interior solution.

2.2 The optimal solution

From the necessary conditions (3) and (4), and the equation of motion for the stock s, one obtains the following system of differential equations for an optimal solution:[2]

$$\begin{aligned} \dot{e}(t) &= \frac{F_e(t)}{F_{ee}(t)}(\gamma+\rho) + \frac{F_s(t+\sigma)}{F_{ee}(t)}\exp[-\rho\sigma] + \frac{F_{es}(t)}{F_{ee}(t)}(\gamma s(t) - e(t-\sigma)) \ , \\ \dot{s}(t) &= e(t-\sigma) - \gamma s(t) \ . \end{aligned} \tag{8}$$

Note that $\dot{e}$ and $\dot{s}$ also depend on *advanced* (i. e. at a later time) and on *retarded* (i. e. at an earlier time) variables. Hence, (8) forms a system of *functional differential equations.*[3] Although this system is not analytically soluble in general, some qualitative properties of the solution can be derived.

Stationary state

First, I derive the fixed points $(e^\star, s^\star)$ of the functional differential equation system (8), which are determined by the conditions $\dot{e} = \dot{s} = 0$.

Proposition 1 (Stationary State)
Given the optimization problem (1) together with the curvature properties (6) for the felicity function $F(e,s)$, the fixed points $(e^\star, s^\star)$ are determined by the following system of implicit equations:

$$\begin{aligned} -\frac{F_s(e^\star, s^\star)}{F_e(e^\star, s^\star)} &= (\gamma+\rho)\exp[\rho\sigma] \ , \\ e^\star &= \gamma s^\star \ , \end{aligned} \tag{9}$$

Furthermore, the stationary state $(e^\star, s^\star)$ exists and is unique, if $-\frac{F_s(\gamma s,s)}{F_e(\gamma s,s)}$ is a monotonically increasing function and satisfies the Inada conditions:

- $\lim_{s\to 0} -\frac{F_s(\gamma s,s)}{F_e(\gamma s,s)} = 0$, *and*
- $\lim_{s\to\infty} -\frac{F_s(\gamma s,s)}{F_e(\gamma s,s)} = \infty$.

Proof: *In the Appendix.*

In the following I assume that the maximization problem (1) exhibits a unique stationary state.

[2] Differentiate (3) with respect to t, insert in (4) and solve for e.
[3] For an introduction to functional differential equations see Asea and Zak (1999: section 2) and Gandolfo (1996: chapter 27). A detailed exposition for linear functional differential equations is given in Bellman and Cooke (1963), and Hale (1977).

Optimal dynamic path and local stability analysis

Second, I examine the system dynamics in a neighborhood around the unique stationary state. Therefore, I linearize the system of functional differential equations (8) around the stationary state $(e^\star, s^\star)$:

$$\begin{aligned}\dot{e}(t) &\approx (\gamma+\rho)(e(t)-e^\star) + \frac{F_{es}}{F_{ee}}\exp[-\rho\sigma](e(t{+}\sigma)-e^\star) - \frac{F_{es}}{F_{ee}}(e(t{-}\sigma)-e^\star)+ \\ &\quad \frac{F_{es}}{F_{ee}}(2\gamma+\rho)(s(t)-s^\star) + \frac{F_{ss}}{F_{ee}}\exp[-\rho\sigma](s(t{+}\sigma)-s^\star)+e^\star, \\ \dot{s}(t) &\approx (e(t{-}\sigma)-e^\star)-\gamma(s(t)-s^\star)+s^\star.\end{aligned} \tag{10}$$

Linear functional differential equations are also called *differential-difference equations*. Similar to the case of ordinary linear first-order differential equations, the elementary solutions for e and s are exponential functions, and the general solution is given by the superposition of the elementary solutions

$$e(t) = e^\star + \sum_n e_n \exp[x_n t]\ , \quad s(t) = s^\star + \sum_n s_n \exp[x_n t]\ , \tag{11}$$

where the e_n and s_n denote constants which can (at least in principle) be unambiguously determined by the set of initial conditions s_0, ξ, and the transversality condition (5). The eigenvalues x_n are the roots of the characteristic polynomial $Q(x)$. The characteristic polynomial $Q(x)$ for the system of differential-difference equations (10) is given by the determinant of the Jacobian of (10) minus the identity matrix times x:

$$Q(x) = \begin{vmatrix} \rho{+}\gamma{+}\frac{F_{es}}{F_{ee}}(\mathrm{e}^{\sigma(x-\rho)}-\mathrm{e}^{-\sigma x}){-}x & \frac{F_{es}}{F_{ee}}(\rho{+}2\gamma){+}\frac{F_{ss}}{F_{ee}}\mathrm{e}^{\sigma(x-\rho)} \\ \mathrm{e}^{-\sigma x} & -\gamma{-}x \end{vmatrix}\ . \tag{12}$$

Introducing the following abbreviations

$$X = \frac{F_{es}(e^\star,s^\star)}{F_{ee}(e^\star,s^\star)}\exp[-\rho\sigma]\ , \quad Y = \frac{F_{es}(e^\star,s^\star)}{F_{ee}(e^\star,s^\star)}\ , \quad Z = \frac{F_{ss}(e^\star,s^\star)}{F_{ee}(e^\star,s^\star)}\exp[-\rho\sigma]{+}\gamma(\gamma{+}\rho)\ ,$$

one obtains for the characteristic equation $Q(x) = 0$:

$$0 = x^2 - \rho x - X\exp[\sigma x](x+\gamma) + Y\exp[-\sigma x](x-\rho-\gamma) - Z\ . \tag{13}$$

The characteristic equation reduces to a quadratic equation with one positive and one negative characteristic root for two special cases, i. e. if there is no time lag $\sigma = 0$ or if the partial derivative $F_{es}(e^\star, s^\star) = 0$. In the general case, the characteristic roots are not analytically solvable. Nevertheless, the characteristic equation (13) can be shown to exhibit an infinite number of complex solutions with positive real part and an infinite number of complex solutions with negative real part. As a consequence, in either case the stationary state $(e^\star, s^\star)$ is a saddle point, i. e. for all initial pollution stocks s_0 and all initial paths ξ, there exists a unique optimal path which converges asymptotically towards the stationary state.

Proposition 2 (Optimal system dynamics)
Given the optimization problem (1) together with the curvature properties (6) for the felicity function $F(e,s)$, the stationary state $(e^\star, s^\star)$ is in general a saddle point. In fact, the characteristic equation (13) of the linear approximation around the stationary state has

- *one positive and one negative real solution, if $\sigma = 0$ (if $F_{es}(e^\star, s^\star) > 0$ then in addition the discount rate ρ has to be sufficiently small),*
- *one positive and one negative real solution, if $F_{es}(e^\star, s^\star) = 0$, and*
- *an infinite number of complex solutions with positive real parts, and an infinite number of complex solutions with negative real parts in the general case.*

Proof: *In the Appendix.*

Note that the characteristic equation (13) may exhibit purely imaginary roots (i. e. complex roots with vanishing real parts). In this case the system of differential-difference equations (10) may exhibit so called *limit-cycles*, i. e. the optimal paths oscillate around the stationary state without converging towards or diverging from it. This issue has been extensively addressed in the literature (e. g. Feichtinger et al. 1994, Asea and Zak 1999, Liski et al. 2001 and Wirl 1995, 1999, 2002). As a consequence, this limit-cycles will not be discussed in the following.

2.3 Time lags and cycles

Because of the transversality condition (5), the system dynamics are restricted to the stable hyperplane, which is spanned by the eigenvectors corresponding to the eigenvalues with negative real parts. According to (11), the optimal paths are monotonic (exponentially damped) if the characteristic roots are real ($x_n \in \mathbb{R}$), and cyclical (and exponentially damped) in the case of complex characteristic roots ($x_n \in \mathbb{C}$). Hence, cyclical optimal paths are not a necessary feature of time lagged stock dynamics, but depend upon the functional form of the felicity function F. A sufficient condition for $F_{es}(e^\star, s^\star) = 0$ is that the felicity function is additively separable.

Proposition 3 (Time lags and cycles)
Given the optimization problem (1) together with the curvature properties (6) for the felicity function $F(e,s)$, the characteristic equation (13) of the linear approximation around the stationary state has exactly one positive and one negative real solution, if the felicity function $F(e,s)$ is additive separable in e and s.

Proof: *F is additively separable*

$$\Rightarrow\ F(e(t), s(t)) = F_1(e(t)) + F_2(s(t)) \ ,$$

$$\Rightarrow\ F_{es}(e,s) = F_{se}(e,s) \equiv 0 \ .$$

Thus, the functional form of the felicity function F plays a crucial role for the qualitative properties of the optimal path. In most (time lagged) capital accumulation models the felicity function F is given by an instantaneous welfare function V which depends upon consumption c

$$F = V(c(t)) \ , \quad c(t) = P(k(t)) - i(t) \ , \tag{14}$$

where i denotes investment, k is the capital stock and P denotes the production function. As V is supposed to exhibit standard curvature properties, i. e. $V' > 0$ and $V'' < 0$, additive separability of F is not assured in time lagged capital accumulation models. Hence, time lagged capital accumulation models in general exhibit cyclical optimal paths as shown in Rustichini (1989) and Asea and Zak (1999).

In fact, the situation is different in many environmental economic problems, where there is assumed additive separability between the welfare gain due to consumption C (which might depend on the amount of emissions e in case of emission abatement) and the welfare loss due to environmental damage D (which depends on the pollution stock s):

$$F = C(e(t), t) - D(s(t)) \ . \tag{15}$$

The additively separable form is often justified by the assumption that welfare losses due to environmental damage D are not too important compared with welfare gains due to increased consumption C (as noted above).

3 General versus additively separable felicity functions

Above, the general properties of optimal control problems with time lags have been analyzed. In the following, I study the optimal path of emissions, which accumulate with a time lag to a pollution stock, within a simple stock accumulation model. Although the model has been inspired by the CFC problem, it is applicable to various stock pollutants. To show the difference between general and additively separable felicity functions in time lagged optimal control problems, the analysis is carried out twice: first with the original not additively separable felicity function, and then with an approximated additively separable felicity function. Both control problems have the formal structure of the generic problem discussed in the previous section.

3.1 The model

Consider an economy with one non-producible input of production (e. g. labor) which is given in a constant maximal amount λ. Furthermore, labor is supposed to be the sole input of the two available production processes in the economy. The first production process produces a consumption good c with constant returns to labor

$$c(t) = l_1(t) \ , \tag{16}$$

where l_1 denotes the amount of labor employed to the consumption good production. In addition, the production of each unit of consumption good gives rise to one unit of gross emissions e^{gross}:

$$e^{gross}(t) = c(t) = l_1(t) \ . \tag{17}$$

The second production process is an abatement process which reduces net emissions e

$$e(t) = e^{gross}(t) - a(t) \ , \tag{18}$$

where a denotes the amount of emissions abated. Denoting the amount of labor employed to the abatement process by l_2, the amount of emissions abated a reads:

$$a(t) = \sqrt{\alpha l_2(t)} \ , \quad \alpha > 0 \ . \tag{19}$$

Note that the second process exhibits decreasing returns to labor. According to the following equation of motion, the net emissions e accumulate with a time lag of the constant and exogenously given size σ to the pollution stock s, which itself decays at a constant rate γ:[4]

$$\dot{s}(t) = e(t-\sigma) - \gamma s(t) \ , \quad \sigma, \gamma > 0 \ . \tag{20}$$

The time lag amounts to the assumption that the pollutant accumulates at a different place to where it has been emitted, and the transportation process needs time. There are many pollutants besides CFCs which accumulate with a time lag to their release into the natural environment, e. g. nitrate and pesticide run-off from agricultural cultivation which seeps away and accumulates in the groundwater and decreases its quality as drinking water (UNEP 2002).

The stock of pollutant s exhibits a negative external effect on the economy. For example, the stock of pollutant s might reduce the effective labor force l:

$$l(t) = \lambda - \beta s(t)^2 \ , \quad \beta > 0 \ . \tag{21}$$

In the case of CFCs, one might think of an increase in the rate of skin cancer with increasing stock of the pollutant which prevents increasingly more people from working. Note that the pollution stock s exhibits increasing marginal damage.

To close the model, I assume that society seeks to maximize its intertemporal welfare W

$$W = \int_0^\infty V(c(t)) \exp[-\rho t] \, dt \ , \quad \rho > 0 \ , \tag{22}$$

where V denotes a twice differentiable, increasing and concave *instantaneous* welfare function which exhibits standard curvature properties ($V' > 0$, $V'' < 0$), and ρ is the constant and positive discount rate. At the optimum, the effective labor force l will be

[4] In general the decay rate γ is a function of the stock s. For the sake of simplicity I assume a constant decay rate γ.

used up completely at any time, as intertemporal welfare W is strictly increasing in consumption c, and labor contributes to more consumption either directly, if employed in the consumption good process, or indirectly, if employed to the abatement process which enables more consumption in the future. Hence,

$$l(t) = l_1(t) + l_2(t) \ . \tag{23}$$

Using equations (16)–(19), (21) and (23), one obtains that consumption c is a function of the emissions e and the pollutant stock s:

$$c(t) = c(e(t), s(t)) = \frac{1}{2}\left[2e(t) - \alpha + \sqrt{4\alpha\left(\lambda - \beta s(t)^2 - e(t)\right) + \alpha^2}\right] \ . \tag{24}$$

Note that the emissions e are bounded from above:

$$e(t) \leq \lambda - \beta s(t)^2 \ . \tag{25}$$

Fortunately, it is not necessary to explicitly control for this inequality, as it cannot be optimal to emit more than $\lambda - \beta s(t)^2$. Instantaneous consumption decreases for $e(t) > \lambda - \beta s(t)^2$ and, in addition, future consumption also decreases with increasing pollution stock s. Note further that the pollution stock s in the time interval $t \in [0, \sigma)$, which is completely determined by the initial stock s_0 and the initial path ξ, has to be smaller than $\sqrt{\lambda/\beta}$. Otherwise, the total labor force would be annihilated before emission control becomes effective.

3.2 General felicity function

From (22) and (24), one derives the following felicity function F:

$$\begin{aligned} F(e(t), s(t)) &= V(c(e(t), s(t))) \\ &= V\left(\frac{1}{2}\left[2e(t) - \alpha + \sqrt{4\alpha\left(\lambda - \beta s(t)^2 - e(t)\right) + \alpha^2}\right]\right) \ . \end{aligned} \tag{26}$$

To apply the results obtained in section 2, I calculate the partial derivatives of F:

$$\begin{aligned}
F_e &= V_c\left(1 - \frac{\alpha}{\sqrt{4\alpha(\lambda-\beta s(t)^2-e(t))+\alpha^2}}\right) , \\
F_s &= V_c\frac{-2\alpha\beta s(t)}{\sqrt{4\alpha(\lambda-\beta s(t)^2-e(t))+\alpha^2}} , \\
F_{ee} &= -V_c\frac{2\alpha^2}{\sqrt[3]{4\alpha(\lambda-\beta s(t)^2-e(t))+\alpha^2}} + V_{cc}\left(1 - \frac{\alpha}{\sqrt{4\alpha(\lambda-\beta s(t)^2-e(t))+\alpha^2}}\right)^2 , \\
F_{ss} &= -V_c\frac{2\alpha^2\beta[4(\lambda-e(t))+\alpha]}{\sqrt[3]{4\alpha(\lambda-\beta s(t)^2-e(t))+\alpha^2}} + V_{cc}\frac{4\alpha^2\beta^2 s(t)^2}{4\alpha(\lambda-\beta s(t)^2-e(t))+\alpha^2} , \\
F_{es} &= -V_c\frac{4\alpha^2\beta s(t)}{\sqrt[3]{4\alpha(\lambda-\beta s(t)^2-e(t))+\alpha^2}} - V_{cc}\frac{2\alpha\beta s(t)\left[\sqrt{4\alpha(\lambda-\beta s(t)^2-e(t))+\alpha^2}-\alpha\right]}{4\alpha(\lambda-\beta s(t)^2-e(t))+\alpha^2} .
\end{aligned}$$

Note that the present value Hamiltonian is strictly concave because of the strict concavity of the felicity function F (V is strictly concave in c by assumption, and c is strictly concave as $c_{ii} < 0$ and $\det[c_{ij}] > 0$, $i, j = e, s$). Hence, the necessary conditions (3) and (4), together with the transversality condition (5), are sufficient for an optimal solution.

According to proposition 1, the following proposition holds for the stationary state:

Proposition 4 (Stationary state – general felicity function)
Given the optimization problem (1) and the felicity function (26), the unique stationary state $(e^\star, s^\star)$ is determined by the following system of implicit equations:

$$\begin{aligned} \frac{2\alpha\beta s^\star}{\sqrt{4\alpha\left(\lambda - \beta s^{\star\,2} - e^\star\right) + \alpha^2} - \alpha} &= (\gamma + \rho)\exp[\rho\sigma] \ , \\ e^\star &= \gamma s^\star \ , \end{aligned} \tag{27}$$

where $s^\star \in (0, \frac{1}{2}\beta(\sqrt{\gamma^2 + 4\beta\lambda} - \gamma))$. Furthermore, the stationary state $(e^\star, s^\star)$ is, ceteris paribus, higher,

- *the lower are the parameters α and β, and*
- *the higher are the discount rate ρ, the decay rate γ and the time lag σ.*

Proof: *In the appendix.*

The economic interpretation of the comparative static results of proposition 4 is straight forward. The higher is β, the higher is the negative stock externality, and the lower are the stationary state pollution stock and emissions. The higher is the discount rate ρ, the lower is the net present value of future damages, and thus, the higher are the stationary state pollution stock and emissions. The higher is the decay rate γ, the faster is the decay of a unit of today's emissions, and the lower is the net present value of the according damage, which leads to higher stationary state pollution stock and emissions. Because of the discounting of future damages, the net present value of the damage caused by a unit of emissions today is lower, the longer is the time lag σ. Hence, the stationary state pollution stock and emissions are higher, the higher is the time lag σ. The higher is the parameter α, the lower are the labor costs of emission abatement. This has two consequences. First, the lower are the emission costs, the stronger is the incentive to abate more emissions, which leads to a lower stationary state pollution stock. Second, the effective labor force needed for a given consumption and abatement level is lower, the higher is α and, thus, a higher stationary state pollution stock and emissions are tolerated. As the first effect dominates the second effect, the stationary state pollution stock and emissions are lower, the higher is the parameter α.

In contrast to the case of an additively separable felicity function, which is discussed in the following section, it is not possible to solve analytically the system of differential-difference-equations (10), which has been derived by the linear approximation around the stationary state. Nevertheless, according to proposition 3, the characteristic equation (13) has an infinite number of complex solutions with positive real parts, and an infinite number of solutions with negative real parts. Thus, cyclical optimal paths are expected in general. Although an analytical derivation of the optimal paths is not possible, one can numerically solve the problem for given exogenous parameters, as shown in section 4.

3.3 Additive separable felicity function

Now suppose that the labor costs of abatement are small, i. e. $\alpha >> 1$, and the consumption loss due to the stock externality is small compared to overall consumption, i. e. $\beta << 1$. In this case, one may neglect the welfare effects of a variation in consumption. Hence, it is sufficient to minimize the costs (in terms of lost consumption) accruing from emission abatement and the stock externality.

Furthermore, the consumption function (24) can be approximated by a first order Taylor-approximation around $\beta = 0$:

$$c(e(t), s(t)) \approx \frac{1}{2}\left[2e(t) - \alpha + \sqrt{4\alpha\,(\lambda - e(t)) + \alpha^2}\right] - \underbrace{\frac{\alpha}{\sqrt{\alpha^2 + 4\alpha(\lambda - e(t))}}}_{A} \beta s(t)^2 \ . \quad (28)$$

If α is big, as supposed, then α^2 outweighs $4\alpha(\lambda - e(t))$ and A is approximately equal to 1. As the maximal amount of consumption which could be produced without the stock externality equals λ, one obtains for the consumption loss L:

$$L(e(t), s(t)) = \lambda - \frac{1}{2}\left[2e(t) - \alpha + \sqrt{4\alpha\,(\lambda - e(t)) + \alpha^2}\right] + \beta s(t)^2 \ . \quad (29)$$

Neglecting the welfare effects, society minimizes L or maximizes $-L$. Hence, the approximated felicity function F reads:

$$F(e(t), s(t)) = \frac{1}{2}\left[2e(t) - \alpha + \sqrt{4\alpha\,(\lambda - e(t)) + \alpha^2}\right] - \lambda - \beta s(t)^2 \ . \quad (30)$$

Note that F is now additively separable in e and s. Hence, from proposition 3 it is expected that the optimal paths are monotonic.

To apply the results obtained in section 2, I calculate the partial derivatives of F of first and second order:

$$\begin{aligned}
F_e &= 1 - \frac{\alpha}{\sqrt{4\alpha\,(\lambda - e(t)) + \alpha^2}} \ , \\
F_s &= -2\beta s(t) \ , \\
F_{ee} &= -\frac{2\alpha^2}{\sqrt[3]{4\alpha\,(\lambda - e(t)) + \alpha^2}} \\
F_{ss} &= -2\beta, \\
F_{es} &= 0 \ .
\end{aligned}$$

Note that the present value Hamiltonian is strictly concave, because of the strict concavity of the felicity function F ($F_{ii} < 0$ and $\det[F_{ij}] > 0$, $i, j = e, s$). Hence, the necessary conditions (3) and (4), together with the transversality condition (5), are sufficient for an optimal solution.

According to proposition 1, the following proposition holds for the stationary state:

Proposition 5 (Stationary state – additively separable felicity function)
Given the optimization problem (1) and the felicity function (30), the unique stationary state $(e^\star, s^\star)$ *is determined by the following system of implicit equations:*

$$\begin{aligned} \frac{2\beta s^\star}{\sqrt{4\alpha\,(\lambda - e^\star) + \alpha^2} - \alpha} &= (\gamma + \rho)\exp[\rho\sigma]\ , \\ e^\star &= \gamma s^\star\ , \end{aligned} \tag{31}$$

where $s^\star \in (0, \lambda/\gamma)$*. Furthermore, the stationary state* $(e^\star, s^\star)$ *is, ceteris paribus, higher,*

- *the lower is the parameter* β*, and*
- *the higher are the parameter* α*, the discount rate* ρ*, the decay rate* γ *and the time lag* σ*.*

Proof: *In the appendix.*

The comparative static effects of variations in the exogenously given parameters are similar to the case of a general felicity function stated in proposition 4, except for the parameter α. Now the second effect dominates and the pollution stock and emissions are the *higher*, the higher the parameter α. This is an artifact of the approximation $A \approx 1$ in (28), which weakens the first effect.

To derive the qualitative properties of the system dynamics, I analyze the system of functional differential equations in the linear approximation around the stationary state. The following proposition summarizes the result:

Proposition 6 (System dynamics – additively separable felicity function)
Given the optimization problem (1) and the felicity function (30), then the stationary state $(e^\star, s^\star)$ *is a saddle point as the characteristic equation reduces to a quadratic equation with one positive and one negative real solution. In addition, the optimal paths for the emissions and the pollution stock converge monotonically towards the stationary state. The system dynamics in the linear approximation around the stationary state are given by*

$$\begin{aligned} e(t) &= (s_\sigma - s^\star)\frac{\beta\exp[-\rho\sigma]\,[4\alpha(\lambda - e^\star)]^{\frac{3}{2}}}{\alpha^2[x_1 - (\gamma + \rho)]}\exp[x_1 t] + e^\star\ , \\ s(t+\sigma) &= (s_\sigma - s^\star)\exp[x_1 t] + s^\star\ , \end{aligned} \tag{32}$$

where $s_\sigma = s(\sigma)$ *denotes the pollution stock at time* $t = \sigma$ *and* x_1 *is the negative characteristic root of the characteristic equation (13):*

$$x_1 = \frac{1}{2}\left[\rho - \frac{1}{\alpha}\sqrt{4\beta\exp[\rho\sigma]\,[4\alpha(\lambda - e^\star) + \alpha^2]^{\frac{3}{2}} + \alpha^2\rho^2}\right]\ <\ 0\ . \tag{33}$$

Proof: *In the appendix.*

Obviously, the optimal paths for the emissions and the pollution stock converge exponentially towards the stationary state values, i.e. the optimal paths are monotonic. Note that the optimal control path e (and thus also the path of the stock s for $t > \sigma$) does not explicitly depend on the initial path ξ, but only on the value $s(\sigma) = s_\sigma$, which is completely determined by the initial stock s_0 together with the initial path ξ and the equation of motion. Hence, the optimal emission path does not directly depend on the initial path ξ as long as the stock s_σ at time $t = \sigma$ remains unaltered. Therefore, the solution of the maximization problem is identical to the non-time-lagged problem, except for the fact that the stationary state depends on the time lag σ, and the pollution stock reacts with a time lag to variations in emissions.

4 Numerical examples

In this section, I illustrate the results derived in the previous section by numerical simulations of the (non linearized) optimization problem (1), given the felicity functions (26) and (30). The numerical optimizations were derived with the advanced optimal control software package MUSCOD-II (Diehl et al. 2001, Leineweber et al. 2003). To apply MUSCOD-II to time lagged optimization problems, it is necessary to reformulate the problem as shown in Winkler et al. (2004).

The exogenous parameters, the initial values and the instantaneous welfare function V used for the numerical optimization are given in table 1. The parameter values have primarily been chosen so as to illustrate clearly the different effects, and do not necessarily reflect the characteristics of real environmental pollution problems. As it is not possible to optimize numerically over an infinite time horizon, the time horizon τ has been set to 200 years, and all parameters have been chosen so that the system at time $t = 200$ is very close to the stationary state (for a more convenient exposition, the figures show just times up to $t = 100$). To illustrate the qualitative difference between instantaneous and time lagged stock accumulation, four different scenarios have been computed. The first scenario shows the optimal system dynamics for instantaneous stock accumulation, i. e. $\sigma = 0$. The remaining three scenarios all exhibit a time lag $\sigma = 10$ between the emissions of the pollutant and their accumulation to the pollution stock, and differ solely in the initial path ξ. To be able to compare the results for these different initial paths – a constant, a linear, and a cyclical initial path – they have been chosen in such a way that the stock of pollution at time $t = \sigma = 10$ is identical for all three of them ($s(10) = s_\sigma = 13$).

α	β	γ	λ	ρ	τ	s_0	V
1	0.005	0.1	1	0.03	200	10	$\ln[c(t)]$

Table 1: Exogenous parameters, initial values and the instantaneous welfare function V used for the numerical optimization.

From propositions 4 and 5, one can compute the stationary state values for the original and the approximated additively separable felicity function, given the parameter values stated in table 1. For time lag $\sigma = 0$, one derives $(e^\star, s^\star) \approx (0.57, 5.70)$ for the original and $(e^\star, s^\star) \approx (0.53, 5.33)$ for the approximated additively separable case, and for $\sigma = 10$, one derives $(e^\star, s^\star) \approx (0.61, 6.10)$ for the original and $(e^\star, s^\star) \approx (0.64, 6.36)$ for the approximated felicity function. In either case the stationary state values are close together for the original and the approximated felicity function. Hence, one might judge that the additively separable felicity function is a good approximation of the original one.

As figure 1 shows, this is true for instantaneous stock accumulation ($\sigma = 0$). But, as expected from propositions 2 and 3, and shown in figure 2, the optimal paths differ substantially, if the time lag $\sigma > 0$.

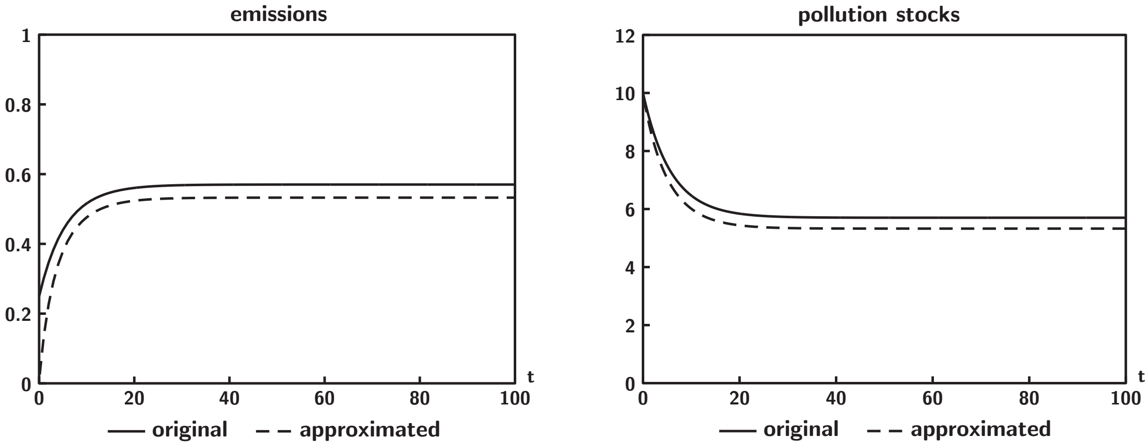

Figure 1: Optimal paths for the emissions (left) and the pollution stock (right) for the original (solid line) and the approximated additively separable (dashed line) felicity function for instantaneous stock accumulation ($\sigma = 0$). Numerical optimization parameters are given in table 1.

Figure 1 shows the optimal paths for the emissions and the pollution stocks in the case of instantaneous stock accumulation ($\sigma = 0$). The emission paths start substantially below their stationary state level of about 0.57 and 0.53 and converge monotonically towards them. Starting from an initial value of 10, the pollution stocks decrease monotonically towards their stationary state values of about 5.70 for the original and 5.33 for the approximated felicity function.

Figure 2 shows the optimal paths of the pollution stocks and the emissions in the case of time lagged stock accumulation ($\sigma = 10$) for three different initial paths ξ (table 2).

Note that the initial paths ξ are also shown as the emission paths in the time interval $t \in [-10, 0]$ in figure 2 and that they are identical for the original and the approximated problem. As already stated in section 2, the path for the pollution stock in the time interval $t \in [0, 10]$ is completely determined by the initial value s_0, the initial path ξ and the equation of motion (20). Hence, it is also identical for the original and the approximated problem. This shows a fundamental feature of time lagged optimal control problems: the system dynamics exhibits an additional moment of inertia as the stock

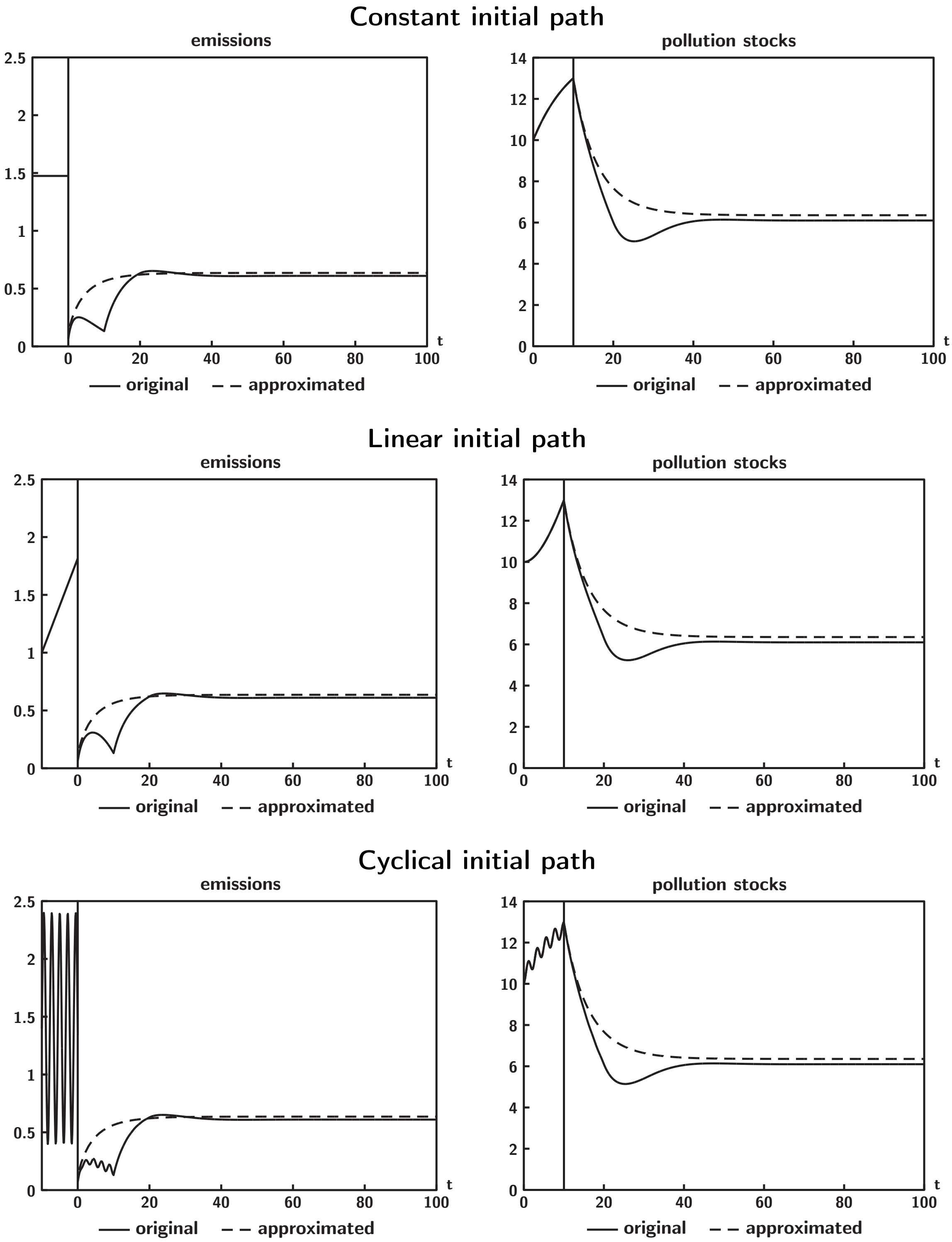

Figure 2: Optimal paths for the emissions (left) and the pollution stock (right) for the original (solid line) and the approximated additively separable (dashed line) felicity function for time lagged stock accumulation ($\sigma = 10$) and three different initial paths ξ. Numerical optimization parameters and initial paths ξ are given in table 1 and table 2.

	constant	linear	cyclical
$\xi(t)$	1.47459	$1 + 0.0815485(t + 10)$	$1.39815 + \sin[0.9\pi(t + 10)]$

Table 2: Initial paths $\xi(t)$ applied to the numerical optimization of the time lagged stock accumulation problem. Note that the initial paths have been chosen so that $s_\sigma = 13$.

reacts delayed to variations in the control. In all three scenarios the pollution stocks rise for times $t < \sigma = 10$ from their initial value $s_0 = 10$ to $s_\sigma = 13$. Nevertheless, because of the different initial paths ξ the path of the pollution stock is concave (ξ constant), convex (ξ linear) or cyclical (ξ cyclical).

In the case of the approximated additively separable felicity function, the emission paths converge monotonically towards their stationary state value, as expected from proposition 6. Note that as $s_\sigma = 13$, no matter which initial path is applied, the optimal emission path is identical in all three cases for the approximated additively separable felicity function. As a consequence, also the optimal paths for the pollution stocks are identical in the approximated case for times $t > \sigma$. Hence, in the case of an additively separable felicity function, variations in the initial path have no influence on the optimal system dynamics as long as the stock s_σ at time $t = \sigma$ remains constant. Overall, if the felicity function is additively separable the system dynamics is qualitatively identical to the case of instantaneous stock accumulation (figure 1).

However, the optimal paths for the original non-additively separable felicity function differ substantially from the optimal paths in the case of an additively separable F in two ways. First, the optimal emission paths are not monotonic and show cyclical behavior. Furthermore, although the optimal emission paths of the original and the approximated problem are close together in the long run, they differ markedly in the short run (especially for times $t < 20$). As a consequence, the optimal paths for the pollution stocks also differ markedly from the optimal path derived from the additively separable approximation. The paths exhibit a pronounced dip between $t = 10$ and $t = 30$, which corresponds to the emissions between $t = 0$ and $t = 20$, because of the time lag σ. Second, if the felicity function is not additively separable, variations in the initial path ξ cause variations in the optimal system dynamics, although the pollution stock $s_\sigma = 13$ remains unaltered. This is best seen in the case of a cyclical initial path, which induces corresponding cycles in the optimal emission path (figure 2 bottom). Overall, although the stationary states of the optimal paths are close together in the long run, there are substantial differences in the optimal paths between the original and the approximated additively separable problem in the short run. Hence, it is questionable if the additively separable felicity function is a justified approximation to the original one.

5 Conclusion

In this paper, I have studied the optimal paths of control problems with one stock and one control variable where the stock reacts with a time lag to a variation in the control.

For a generic control problem, with a time lagged equation of motion, I have shown that a unique saddle point stationary state exists, and that the system dynamics crucially depend on the functional form of the objective function. In general, the system dynamics is characterized by non-monotonic and cyclical paths, as suggested by Rustichini (1989), and Asea and Zak (1999) in the time-to-build literature. Nevertheless, the optimal paths converge monotonically towards the stationary state if the objective function is additively separable in the control and the stock variables.

As shown in an example with time lagged accumulation of a pollution stock, the optimal paths may differ substantially between the original and an approximated additive separable felicity function, although the long-term stationary states to which both systems converge, are quite close together. Furthermore, in the case of a non-additively separable felicity function, the optimal paths also depend on the initial path ξ. If the strong assumptions of perfect information and perfect foresight, which are necessary to justify the optimal control approach, are satisfied, these results question the application of additively separable welfare functions as good approximations to the "real world" in environmental economics. In fact, in the case of time lagged stock accumulation we face a threefold dilemma. First, we have to cope with an additional moment of inertia because of the time lag between the control and the accumulation of stock, which demands increased caution and alertness in the handling of time lagged pollutants as CFCs. Second, the application of an easy to handle additive separable welfare function as a good approximation to the real world problem might be an unjustified oversimplification. Third, non-monotonic and cyclical optimal paths may be very difficult to implement by most political regimes. Thus, even if the optimal emission path is known, it might be not applicable because of institutional constraints.

Overall, the analysis suggests that it is likely that only a non-monotonic control is optimal, if emissions accumulate time lagged to pollution stocks. A similar conclusion has been derived in recent work by Moslener and Requate (2001), Jöst et al. (2003) and Baumgärtner et al. (2003) for *instantaneous accumulation* of *various interacting* pollution stocks. As real world environmental problems are normally characterized by the fact that stocks of pollutants interact and/or accumulate time lagged to the corresponding emissions, this result may be relevant for a large number of environmental problems, such as climate change, depletion of the ozone layer, groundwater contamination, acidification of soil and surface water, biodiversity loss, etc. In all these cases, when formulating policy advice it is important to take account of the history, the empirical parameter values and the complex dynamic relationships of the problem.

Although the results show that there is a *qualitative* difference in the optimal control of instantaneous and time lagged accumulation problems, it is not obvious how the transition from $\sigma = 0$ to $\sigma > 0$ takes place *quantitatively*. This issue is addressed in Winkler et al. (2004). First results suggest that there is a continuous deviation of the optimal path for an continuously increasing time lag σ. Another interesting question, which is open to future research, is the analysis of "second best" *monotonic* optimal paths if the *non-monotonic* first best optimal path is not applicable due to institutional constraints.

Appendix

A.1 Proof of Proposition 1

Implicit Equations

From $\dot{e} = \dot{s} = 0$ one derives the following implicit equations:

$$\begin{aligned} 0 &= \frac{1}{F_{ee}(e^\star, s^\star)} \left(F_e(e^\star, s^\star)(\gamma + \rho) + F_s(e^\star, s^\star) \exp[-\rho\sigma] \right) \ , \\ 0 &= e^\star - \gamma s^\star \ . \end{aligned}$$

Furthermore, $F_{ee}(e^\star, s^\star) \neq 0$, because of the assumed curvature properties (6). Hence, the first equation can be rearranged to yield:

$$-\frac{F_s(e^\star, s^\star)}{F_e(e^\star, s^\star)} = (\gamma + \rho) \exp[\rho\sigma] \ .$$

Existence and Uniqueness

The Inada conditions $\lim_{s\to 0} -\frac{F_s(\gamma s, s)}{F_e(\gamma s, s)} = 0$ and $\lim_{s\to\infty} -\frac{F_s(\gamma s, s)}{F_e(\gamma s, s)} = \infty$ together with the continuity of F_e and F_s guarantee the existence of at least one $(e^\star, s^\star)$ which satisfies the equations (9). If $-\frac{F_s(\gamma s, s)}{F_e(\gamma s, s)}$ is a monotonically increasing function in s then there is at most one $(e^\star, s^\star)$ which satisfies the equations (9) and the uniqueness is guaranteed.

A.2 Proof of Proposition 2

Case 1: $\sigma = 0$

If the time lag $\sigma = 0$, the characteristic equation (13) reduces to the quadratic equation

$$x^2 - \rho x - \gamma(\gamma + \rho) - \frac{F_{es}(e^\star, s^\star)}{F_{ee}(e^\star, s^\star)}(2\gamma + \rho) - \frac{F_{ss}(e^\star, s^\star)}{F_{ee}(e^\star, s^\star)} = 0 \ ,$$

with the solutions:

$$x_{1/2} = \frac{\rho}{2} \pm \sqrt{\left(\frac{\rho}{2} + \gamma\right)^2 + \frac{F_{ss}(e^\star, s^\star)}{F_{ee}(e^\star, s^\star)} + \frac{F_{es}(e^\star, s^\star)}{F_{ee}(e^\star, s^\star)}(2\gamma + \rho)} \ . \tag{A.1}$$

If $F_{es}(e^\star, s^\star) \leq 0$ then $x_1 > 0$ and $x_2 < 0$. If $F_{es}(e^\star, s^\star) > 0$ then the square root can be shown to be positive because of the assumed curvature properties (6). Hence, if the discount rate ρ is sufficiently small then $x_1 > 0$ and $x_2 < 0$.[5]

[5] If $F_{es}(e^\star, s^\star) > 0$ then a diverging path for e and s can be optimal (if both roots are positive). Note that in this case at least one root is smaller than ρ, hence the transversality condition (5) will still be satisfied.

Case 2: $F_{es}(e^\star, s^\star) = 0$

For $F_{es}(e^\star, s^\star) = 0$ the characteristic equation 13 reduces to the quadratic equation

$$x^2 - \rho x - \frac{F_{ss}(e^\star, s^\star)}{F_{ee}(e^\star, s^\star)} \exp[-\sigma\rho] - \gamma(\gamma + \rho) \ ,$$

with the solutions:

$$x_{1/2} = \frac{\rho}{2} \pm \sqrt{\rho^2 + \frac{F_{ss}(e^\star, s^\star)}{F_{ee}(e^\star, s^\star)} \exp[-\rho\sigma] + \gamma(\gamma + \rho)} \ . \tag{A.2}$$

According to the assumed curvature properties (6) $F_{ss}/F_{ee} > 0$. Hence, the characteristic equation exhibits a unique positive and a unique negative real solution.

Case 3: $\sigma > 0 \ , F_{es}(e^\star, s^\star) \neq 0$

In the general case, the characteristic roots are not analytically solvable. Nevertheless, the characteristic equation (13) can be shown to exhibit an infinite number of complex solutions with positive real part and an infinite number of complex solutions with negative real part. To see this, insert $x = a + ib$ with $a, b \in \mathbb{R}$ into the characteristic equation, and separate the real and complex parts:

$$\begin{aligned}
0 &= [Y \exp[-\sigma a](a - \gamma - \rho) - X \exp[\sigma a](a + \gamma)] \cos[\sigma b] \\
&\quad + b \,[Y \exp[-\sigma a] + X \exp[\sigma a]] \sin[\sigma b] + a^2 - b^2 - a\rho - Z \ , \\
0 &= b(2a - \gamma) + b \,[Y \exp[-\sigma a] - X \exp[\sigma a]] \cos[\sigma b] \\
&\quad + [Y \exp[-\sigma a](\gamma + \rho - a) X \exp[\sigma a](a + \gamma)] \sin[\sigma b] \ .
\end{aligned}$$

Equalling these equations, after solving both for $\sin[\sigma b]$, yields

$$0 = b\left(\chi_1(a) + \chi_2(a)\cos[\sigma b]\right) + \frac{1}{b}\left(\chi_3(a) + \chi_4(a)\cos[\sigma b]\right) \ , \tag{A.3}$$

with the following functions χ_i $(i = 1, \ldots, 4)$:

$$\chi_1 = \frac{X \exp[3\sigma a](a-\gamma-\rho) + Y \exp[\sigma a](a+\gamma)}{Y^2(\rho+\gamma-a) + XY \exp[2\sigma a](\rho-2a) - X^2 \exp[4\sigma a](a+\gamma)} \ , \quad \chi_3 = \frac{\exp[\sigma a](a(a+X+\rho) - Z - a\gamma)}{X \exp[2\sigma a] + Y} \ ,$$

$$\chi_2 = \frac{Y^2 - X^2 \exp[4\sigma a]}{Y^2(\rho+\gamma-a) + XY \exp[2\sigma a](\rho-2a) - X^2 \exp[4\sigma a](a+\gamma)} \ , \quad \chi_4 = \frac{X \exp[2\sigma a](a-\gamma) - Y(a+\gamma+\rho)}{X \exp[2\sigma a] + Y} \ .$$

As the right summand of equation (A.3) converges asymptotically towards 0 for large b, this equation has at least one solution, if $|\chi_1| \leq |\chi_2|$ for some given a. As $\exp[4\sigma a]$ in χ_2 is stronger increasing in a as $a \exp[3\sigma a]$ in χ_1, there exits a lower bound $\overline{a} > 0$ with $a > \overline{a} \Rightarrow |\chi_1| \leq |\chi_2|$. Furthermore, as χ_1 converges towards 0 and χ_2 towards Y^2 for $a \to -\infty$, there exists an upper bound $\underline{a} < 0$ with $a < \underline{a} \Rightarrow |\chi_1| \leq |\chi_2|$. Hence, the characteristic equation (13) has an infinite number of complex solutions with positive, and an infinite number of complex solutions with negative real part.

A.3 Proof of proposition 4

The implicit equations 27 follow directly from proposition 1. The stationary state is unambiguously determined by these equations, as $-\frac{F_s}{F_e}(s, \gamma s)$ is a monotonically increasing function in s and may reach any positive value for $s \in (0, \frac{1}{2}\beta[\sqrt{\gamma^2 + 4\beta\lambda} - \gamma])$.

To show the comparative static properties of the stationary state, I insert the second equation of (27) into the first one and define the following function S:

$$S = \frac{2\alpha\beta s^\star}{\sqrt{4\alpha\left(\lambda - \beta s^{\star 2} - \gamma s^\star\right) + \alpha^2} - \alpha} - (\gamma + \rho)\exp[\rho\sigma] \ = 0 \ .$$

Applying the total differential of S, one derives for the first derivatives of $s^\star$:

$$\begin{aligned}
&\frac{ds^\star}{d\alpha} = -\frac{\partial S/\partial\alpha}{\partial S/\partial s^\star} < 0 \ , \quad \frac{ds^\star}{d\beta} = -\frac{\partial S/\partial\beta}{\partial S/\partial s^\star} < 0 \ , \\
&\frac{ds^\star}{d\gamma} = -\frac{\partial S/\partial\gamma}{\partial S/\partial s^\star} > 0 \ , \quad \frac{ds^\star}{d\rho} = -\frac{\partial S/\partial\rho}{\partial S/\partial s^\star} > 0 \quad \frac{ds^\star}{d\sigma} = -\frac{\partial S/\partial\sigma}{\partial S/\partial s^\star} > 0 \ .
\end{aligned}$$

A.4 Proof of proposition 5

The implicit equations (27) follow directly from proposition 1. The stationary state is unambiguously determined by these equations as $-\frac{F_s}{F_e}(s, \gamma s)$ is a monotonically increasing function in s, $\lim_{s\to 0} -\frac{F_s}{F_e}(s, \gamma s) = 0$, and $\lim_{s\to\lambda/\gamma} -\frac{F_s}{F_e}(s, \gamma s) = \infty$.

To show the comparative static properties of the stationary state, I insert the second equation of (31) into the first one and define the following function S:

$$S = \frac{2\beta s^\star}{\sqrt{4\alpha\left(\lambda - \gamma s^\star\right) + \alpha^2} - \alpha} - (\gamma + \rho)\exp[\rho\sigma] \ = 0 \ .$$

Applying the total differential of S, one derives for the first derivatives of $s^\star$:

$$\begin{aligned}
&\frac{ds^\star}{d\alpha} = -\frac{\partial S/\partial\alpha}{\partial S/\partial s^\star} > 0 \ , \quad \frac{ds^\star}{d\beta} = -\frac{\partial S/\partial\beta}{\partial S/\partial s^\star} < 0 \ , \\
&\frac{ds^\star}{d\gamma} = -\frac{\partial S/\partial\gamma}{\partial S/\partial s^\star} > 0 \ , \quad \frac{ds^\star}{d\rho} = -\frac{\partial S/\partial\rho}{\partial S/\partial s^\star} > 0 \quad \frac{ds^\star}{d\sigma} = -\frac{\partial S/\partial\sigma}{\partial S/\partial s^\star} > 0 \ .
\end{aligned}$$

A.5 Proof of proposition 6

According to (A.1), the solutions of the characteristic equation (13) read:

$$\begin{aligned}
x_1 &= \frac{1}{2}\left[\rho - \frac{1}{\alpha}\sqrt{4\beta\exp[\rho\sigma]\left[4\alpha(\lambda - e^\star) + \alpha^2\right]^{\frac{3}{2}} + \alpha^2\rho^2}\right] \ < \ 0 \ , \\
x_2 &= \frac{1}{2}\left[\rho + \frac{1}{\alpha}\sqrt{4\beta\exp[\rho\sigma]\left[4\alpha(\lambda - e^\star) + \alpha^2\right]^{\frac{3}{2}} + \alpha^2\rho^2}\right] \ > \ 0 \ .
\end{aligned}$$

Hence, the system dynamics in the linear approximation around the stationary state is given by (e. g. Gandolfo 1996: 242):

$$
\begin{aligned}
e(t) &= c_1 \exp[x_1 t] + c_2 \exp[x_2 t] + e^\star \ , \\
s(t) &= c_1 \frac{x_1 - q_{11}}{q_{12}} \exp[x_1 t] + c_2 \frac{x_1 - q_{11}}{q_{12}} \exp[x_2 t] + s^\star \ ,
\end{aligned}
\tag{A.4}
$$

where c_1 and c_2 denote constants which are determined by the initial conditions s_0, ξ and the transversality condition (5), and q_{11}, q_{12} are the matrix elements in the first line of the characteristic polynomial (12). Because of the transversality condition (5), the optimal path is restricted to the stable hyperplane which is spanned by the eigenvectors associated with the negative eigenvalues, i. e. $c_2 = 0$. Hence, (A.4) reduces to:

$$
\begin{aligned}
e(t) &= c_1 \exp[x_1 t] + e^\star \ , \\
s(t) &= c_1 \frac{\alpha^2 [x_1 - (\gamma + \rho)]}{\beta \exp[-\rho\sigma] \, [4\alpha(\lambda - e^\star)]^{\frac{3}{2}}} \exp[x_1 (t - \sigma)] + s^\star \ ,
\end{aligned}
\tag{A.5}
$$

To derive c_1, one calculates the pollution stock at time $t = \sigma$ which is determined by the initial stock s_0, the initial path ξ and the equation of motion (20). Denoting $s(\sigma) = s_\sigma$ and inserting in (A.5), yields:

$$
s_\sigma = c_1 \frac{\alpha^2 [x_1 - (\gamma + \rho)]}{\beta \exp[-\rho\sigma] \, [4\alpha(\lambda - e^\star)]^{\frac{3}{2}}} + s^\star \ .
$$

Thus, c_1 equals:

$$
c_1 = (s_\sigma - s^\star) \frac{\beta \exp[-\rho\sigma] \, [4\alpha(\lambda - e^\star)]^{\frac{3}{2}}}{\alpha^2 [x_1 - (\gamma + \rho)]} \ .
$$

Hence, the system dynamics in the linear approximation around the stationary state is given by:

$$
\begin{aligned}
e(t) &= (s_\sigma - s^\star) \frac{\beta \exp[-\rho\sigma] \, [4\alpha(\lambda - e^\star)]^{\frac{3}{2}}}{\alpha^2 [x_1 - (\gamma + \rho)]} \exp[x_1 t] + e^\star \ , \\
s(t + \sigma) &= (s_\sigma - s^\star) \exp[x_1 t] + s^\star \ .
\end{aligned}
$$

References

Aaheim, H. (1999): 'Climate policy with multiple sources and sinks of greenhouse gases', *Environmental and Resource Economics*, **14:** 413–29.

Asea, P. K. and P. J. Zak (1999): 'Time-to-build and cycles', *Journal of Economic Dynamics and Control*, **23:** 1155–75.

Baumgärtner, S., Jöst, F. and R. Winkler (2003): 'Optimal dynamic scale and structure of a multi-pollution economy'. *Discussion-Paper No. 398*, Alfred Weber-Institute of Economics, University of Heidelberg.

Bellman, R. and K. L. Cooke (1963): *Differential-Difference Equations.* Academic Press, New York.

Diehl, M., Leineweber, D. B. and A. A. S. Schäfer (2001): 'Muscod-II users' manual'. *Preprint 2001-25*, Interdisciplinary Center for Scientific Computing, University of Heidelberg.

El-Hodiri, M. A., Loehman, E. and A. Whinston (1972): 'An optimal growth model with time lags', *Econometrica*, **40:** 1137–46.

Faber, M., Proops, J. L. R. and S. Speck (1999): *Capital and Time in Ecological Economics.* Edward Elgar, Cheltenham.

Falk, I. and R. Mendelsohn (1993): 'The economics of controlling stock pollutants: An efficient strategy for greenhouse gases', *Journal of Environmental Economics and Management*, **25:** 76–88.

Feichtinger, G., Novak, A. and F. Wirl (1994): 'Limit cycles in intertemporal adjustment models: Theory and application', *Journal of Economic Dynamics and Control*, **18:** 353–18.

Forster, B. (1973): 'Optimal capital accumulation in a polluted environment', *Southern Economic Journal*, **39:** 544–7.

Gandolfo, G. (1996): *Economic Dynamics.* Third, completely revised and enlarged edition. Springer, Berlin.

Gradus, R. and S. Smulders (1993): 'The trade-off between environmental care and long-term growth', *Journal of Economics*, **58:** 25–51.

Gradus, R. and S. Smulders (1996): 'Pollution abatement and long-term growth', *European Journal of Political Economy*, **12:** 506–32.

Hale, J. (1977): *Theory of Functional Differential Equations.* Springer, New York.

Ioannides, Y. M. and B. Taub (1992): 'On dynamics with time-to-build investment and non-separable leisure', *Journal of Economic Dynamics and Control*, **16:** 225–41.

Jöst, F., Quaas, M. and J. Schiller (2003): 'Population growth and environmental deterioration: An intertemporal perspective'. *Discussion-Paper No. 400*, Alfred Weber-Institute of Economics, University of Heidelberg.

Kalecki, M. (1935): 'A macroeconomic theory of business cycles', *Econometrica*, **3:** 327–44.

Keeler, E. M., Spence, E. M. and R. Zeckhauser (1972): 'The optimal control of pollution', *Journal of Economic Theory*, **4:** 19–34.

Kydland, F. E. and E. C. Prescott (1982): 'Time to build and aggregate fluctuations', *Econometrica*, **50:** 1345–70.

Leineweber, D. B., Bauer, I., Bock, H. G. and J. P. Schlöder (2003): 'An efficient multiple shooting based reduced SQP strategy for large-scale dynamic process optimization – part I: Theoretical aspects', *Computers and Chemical Engineering*, **27:** 157–66.

Liski, M., Kort, P. M. and A. Novak (2001): 'Increasing returns and cycles in fishing', *Resource and Energy Economics*, **23:** 241–58.

Michaelis, P. (1999): 'Sustainable greenhouse policies: The role of non-CO_2 gases', *Structural Change and Economic Dynamics*, **10:** 239–60.

Moslener, U. and T. Requate (2001): 'Optimal abatement strategies for various interacting greenhouse gases – is the global warming potential a useful indicator'. *Discussion-Paper No. 360*, Alfred Weber-Institute of Economics, University of Heidelberg.

Van der Ploeg, F. and C. Withagen (1991): 'Pollution control and the Ramsey problem', *Environmental and Resource Economics*, **1:** 215–30.

Plourde, C. (1972): 'A model of waste accumulation and disposal', *Canadian Journal of Economics*, **5:** 119–25.

Rustichini, A. (1989): 'Hopf bifurcation for functional differential equations of mixed type', *Journal of Dynamics and Differential Equations*, **1:** 145–77.

Smith, V. L. (1977): 'Control theory applied to natural and environmental resources – an exposition', *Journal of Environmental Economics and Management*, **4:** 1–24.

Winkler, R. (2002): 'Structural change with joint production of consumption and environmental pollution: A neo-austrian approach'. *Discussion-Paper No. 386*, Alfred Weber-Institute of Economics, University of Heidelberg.

Winkler, R., Brand-Pollmann, U., Moslener, U. and J. Schlöder (2004): 'Qualitative and quantitative impact of time-to-build in optimal control models'. Non published manusscript, University of Heidelberg, January 2004.

Wirl, F. (1995): 'The cyclical exploitation of renewable resource stocks may be optimal', *Journal of Environmental Economics and Management*, **29:** 252–61.

Wirl, F. (1999): 'Complex, dynamic environmental policies', *Resource and Energy Economics*, **21:** 19–41.

Wirl, F. (2002): 'Stability and limit cycles in competitive equilibria subject to adjustment costs and dynamic spillovers', *Journal of Economic Dynamics and Control*, **26:** 375–98.