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Abstract. It is shown that the thermodynamic law of conservation of mass, the so-called Materials-Balance-Principle, implies that the marginal product as well as the average product of a material resource input are bounded from above. This means that the Inada conditions, one of the standard assumptions of economic growth theory, when applied to material resource inputs are inconsistent with a basic law of nature. The analysis is based on a model of multi-level production where intermediate goods are produced from elementary resources, and an all-purpose final commodity is produced from these intermediates.

JEL-classification: E13, O40, Q30

Key words: conservation of mass, Inada conditions, materials balance principle, natural resources, production function, thermodynamics

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1 Introduction

It is characteristic for many of the pioneering theoretical contributions to the analysis of economic growth under scarcity of exhaustible natural resources (Dasgupta and Heal 1974, Hoel 1978, Mäler 1974, Schulze 1974, Smith 1977, Solow 1974, Stiglitz 1974, Weinstein and Zeckhauser 1974) as well as large parts of the vast strand of literature which they initiated that they take the well established neoclassical growth theory as their starting point and extend it in the most simple way to also include natural resources, namely by adding one additional variable representing material resource input into a neoclassical aggregate production function. This production function is usually assumed to display the standard properties concerning the resource factor (substitutability between factors of production, positive decreasing marginal product approaching zero and infinity in the two limits of infinite and vanishing resource input).[2]

However, this procedure does not appropriately account for the fact that the extraction of material resource inputs, their transformation within the production process, and their emission or disposal after use are, at root, transformations of energy and matter. As such they are subject to the laws of thermodynamics, which is the branch of physics dealing with transformations of energy and matter. Thermodynamic relations, thus, may impose additional constraints on economic action (Daly 1997b, Solow 1997).

This paper formally explores one particular implication that the thermodynamic law of conservation of mass, the so-called Materials-Balance-Principle, has for modeling production. It is shown that the marginal product as well as the average product of a material resource input may be bounded from above. This means that the usual Inada conditions (Inada 1963), when applied to material resource inputs, can be inconsistent with a basic law of nature. This is im-

[2]Historically, the concept of a production function has been introduced by Wicksell (1893) and Wicksteed ([1894]1992) to analyse the *distribution* of income among the factor owners, and not its *physical production* (Schumpeter 1954: 1028, Sandelin 1976). But later on, the concept has come to dominate economists' thinking about the physically feasible production possibilities, as in the discussion of economic growth under scarcity of natural resources.

portant since the Inada conditions are usually held to be crucial for establishing steady state growth under scarce exhaustible resources. While the advocates of a thermodynamic-limits-to-economic-growth perspective (e.g. Boulding 1966, Georgescu-Roegen 1971, Daly 1977/1991) usually stress the universal and inescapable nature of limits imposed by laws of nature, pro-economic-growth advocates usually claim that there is plenty of scope for getting around particular thermodynamic limits by substitution, technical progress and "dematerialization" (e.g. Beckerman 1999, Smulders 1999, Stiglitz 1997). The latter therefore often conclude that, on the whole, thermodynamic constraints are simply irrelevant for economics. This paper takes a more differentiated stand, by analyzing in detail

(i) what exactly are the implications of thermodynamics for modeling production at the level of a single production process, and

(ii) how these constraints carry over to the level of aggregate production, considering that there is scope for substitution in an economy between different resources and different production technologies.

This paper continues and merges two strands in the literature on the production function. In a first strand, the neoclassical production function has been critically discussed against the background of thermodynamics. Georgescu-Roegen (1971) claims that the neoclassical production function is incompatible with the laws of thermodynamics, basically because it does not properly reflect the irreversible nature of transformations of energy and matter, and because it confounds flow and fund quantities (Daly 1997a, Kurz and Salvadori 2003). Berry et al. (1978) and Dasgupta and Heal (1979: Chap. 7) demonstrate that the conservation laws for energy and matter imply that substitutability between energy-matter inputs, which are subject to the laws of thermodynamics, and other inputs such as labor or capital, which lie outside the domain of thermodynamics, is restricted. All these apparent inconsistencies between the laws of thermodynamics and the standard assumptions about the neoclassical production function have led to more general descriptions of the production process, which blend the traditional theory of production with the thermodynamic principle of conservation of mass (Anderson 1987, Baumgärtner 2000: Chap. 4, Pethig 2003: Sec. 3.3).

Another strand in the literature, more narrowly concerned with production theory (Shephard 1970), has focused specifically on the Inada conditions. It has been demonstrated that the Inada conditions follow from other basic properties of the neoclassical production function (Dyckhoff 1983), and that they impose strong restrictions on the asymptotic behavior of the elasticity of substitution between capital and labor (Barelli and de Abreu Pessôa 2003). Furthermore, the Inada conditions have been shown to be incompatible with another basic principle within economics, the Law of Diminishing Returns (Färe and Primont 2002).

In section 2, the Inada conditions are briefly reviewed in the context of neoclassical growth theory with and without natural resources. Section 3 provides a thermodynamic analysis of the production process at the micro level, i.e. for a micro production function for a single commodity. Section 4 explores the implications for the Inada conditions at the macro-level, i.e. for an aggregate production function for an all purpose commodity. Section 5 concludes.

2 The Inada conditions on resource inputs

With just two inputs, capital K and labor L, the aggregate neoclassical production function for output Y takes the form $Y = F(K, L)$. It is usually assumed to exhibit constant returns to scale and positive and diminishing marginal products with respect to each input for all $K, L > 0$ (Solow 1956, Swan 1956):

$$\frac{\partial F}{\partial K} > 0, \quad \frac{\partial^2 F}{\partial K^2} < 0, \quad \frac{\partial F}{\partial L} > 0, \quad \frac{\partial^2 F}{\partial L^2} < 0. \tag{1}$$

Furthermore, following Inada (1963) the marginal product of an input is assumed to approach infinity as this input goes to zero and to approach zero as the input goes to infinity:

$$\lim_{K \to 0} \frac{\partial F}{\partial K} = \lim_{L \to 0} \frac{\partial F}{\partial L} = +\infty, \qquad \lim_{K \to +\infty} \frac{\partial F}{\partial K} = \lim_{L \to +\infty} \frac{\partial F}{\partial L} = 0. \tag{2}$$

In growth models these so-called Inada conditions are crucial for the existence of interior steady state growth paths: they are sufficient (yet not necessary) for the existence of an interior solution in which the economy grows at a constant and strictly positive rate. Assumptions (1) and (2) imply that each input is essential for production, that is, $F(0, L) = F(K, 0) = 0$, and that output goes to infinity as either input goes to infinity.

When extending the framework of neoclassical growth theory to also include natural resources this is usually done by including one additional variable, R, into the production function, representing material resource input: $Y = F(K, L, R)$ (Dasgupta and Heal 1974: 9, Solow 1974: 34, Stiglitz 1974: 124). The same standard assumptions are then made about this resource dependent production function F as made before about the capital-labor-only-production function. For instance, F is assumed to be increasing, strictly concave, twice differentiable, and linear homogeneous (Dasgupta and Heal 1974: 9, Solow 1974: 34, Stiglitz 1974: 124). Furthermore, some more or less direct analogue to the Inada conditions is assumed in order to guarantee existence of non-trivial (interior) solutions. For example, Solow (1974: 34) assumes that resources are essential for production, i.e. $F(K, L, 0) = 0$, and, at the same time the average product of R has no upper bound, i.e. there is no $\alpha < +\infty$ with $F/R < \alpha$. While this is a particular form of the Inada condition, since it necessarily follows from $\lim_{R\to 0} \partial F/\partial R = +\infty$, Dasgupta and Heal (1974: 11) directly assume that $\lim_{R\to 0} \partial F/\partial R = +\infty$.

Based on one or the other form of Inada conditions, the result is that even with a limited reservoir of an exhaustible natural resource and with that resource being essential for production it is possible to maintain a positive and constant level of consumption forever (Solow 1974). If there is technical progress there might even be exponentially growing consumption (Stiglitz 1974). And while the remaining stock of the resource will approach zero along the optimal path, the resource will never completely be exhausted (Dasgupta and Heal 1974).

So, in a sense, these analyses seem to have produced a rather optimistic answer to the "Limits to growth"-concern (Meadows et al. 1972). However, the Inada conditions as applied (in whatever form) to material resource inputs may be inconsistent with the thermodynamic law of conservation of mass. This is demonstrated in the following.

3 Thermodynamic limits to resource productivity at the micro level

The First Law of Thermodynamics implies that matter can neither be created nor annihilated, i.e. in a closed system it is conserved.[3] This law establishes what is known as 'Materials-Balance-Principle' in environmental and resource economics (Ayres 1999, Pethig 2003).

In order to infer this Law's implications for the production process, consider the following simple model of production at the micro level, i.e. production of a particular good by a particular elementary technology. For the moment assume that there is only one single natural resource.[4] Production of output Y depends – besides capital K and labor L – on the resource material R:

$$Y = F(K, L, R). \tag{3}$$

As a by-product the production process yields the non-negative amount W of waste. All three, R, Y and W, are measured in physical (mass) units. Let ρ with $0 < \rho \leq 1$ denote the (mass) fraction of resource material contained in the output, and μ with $0 \leq \mu \leq 1$ the (mass) fraction of resource material contained in the waste.[5] While ρ is, in general, a function of K and L, i.e. the resource content of the final product may be decreased by using more capital and labor ("dematerialization"), there obviously are physical limits to dematerialization. For instance, in order to produce one kilogram of iron screws one needs to employ at least one kilogram of pure iron. This means that ρ is physically bounded from below, in particular $\rho > 0$. Therefore, one may take ρ as a constant denoting the lower bound to dematerialization, i.e. $\rho = const.$ with $0 < \rho \leq 1$ denotes the

[3] A *closed* thermodynamic system is one that does not exchange matter with its surrounding. It may, however, exchange energy with its surrounding. For an elementary introduction into thermodynamics, see e.g. Kondepudi and Prigogine (1998).

[4] The generalization to many different natural resources will be done in section 4 below.

[5] That ρ and μ are allowed to be less than 1 is due to materials other than the natural resource R considered here. These other materials might enter the production process and be part of the product as well as of the waste. Yet, they are not explicitly represented here.

minimum (mass) fraction of resource material contained in the output.

Applying the materials-balance-principle to the production process results in the following formal balance equation:

$$R = \rho F(K, L, R) + \mu W, \tag{4}$$

which states that the resource material which enters the process also eventually has to come out of the process, be it in the desired product or in the waste. Rearranging equation (4) into

$$\frac{F(K, L, R)}{R} = \frac{1}{\rho}\left(1 - \frac{\mu W}{R}\right)$$

and noting that $\mu W/R \geq 0$ yields the following upper bound for the average resource product F/R:

$$\frac{F(K, L, R)}{R} \leq \frac{1}{\rho}. \tag{5}$$

This establishes the following result.

PROPOSITION 1: *The average product of resource input, $F(K, L, R)/R$, is bounded from above by the inverse of the resource fraction in the good produced, $1/\rho$.*

Proposition 1 has the following implication for the shape of the production function. Equation (5) can be rearranged into

$$F(K, L, R) < \frac{1}{\rho} R,$$

which states that for fixed values of K and L the graph of F plotted as a function of R always stays below a line of slope $1/\rho$ starting at the origin (figure 1). As ρ becomes smaller, the upper limit on the average resource product will grow. However, the upper limit will always remain finite, since ρ is strictly positive.

With the *average* resource product F/R bounded from above by the inverse of the resource fraction in the good produced, $1/\rho$, a similar argument applies to the *marginal* resource product. Taking the total differential of the material balance equation (4) and considering only variations in R, i.e. $dK = dL = 0$, yields

$$dR = \rho \frac{\partial F(K, L, R)}{\partial R} dR + \mu \frac{\partial W(K, L, R)}{\partial R} dR \; .$$

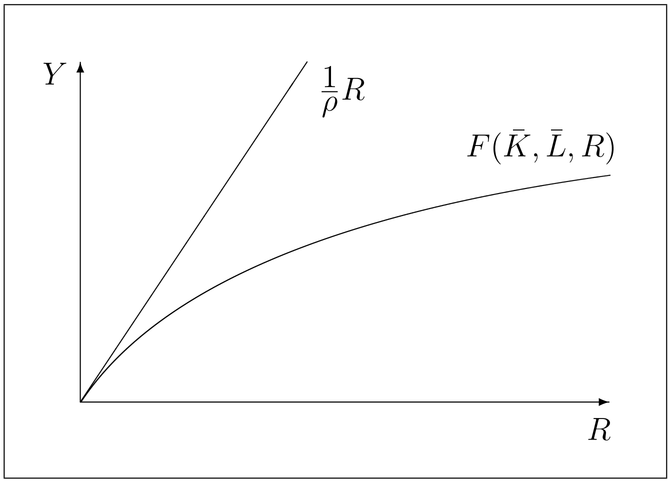

Figure 1: The materials-balance-principle implies that the graph of $F(\bar{K},\bar{L},R)$ is bounded from above by a line of slope $1/\rho$ starting at the origin.

This holds for all $dR \geq 0$ and, thus, implies

$$1 = \rho \, \frac{\partial F(K,L,R)}{\partial R} + \mu \, \frac{\partial W(K,L,R)}{\partial R} \; . \tag{6}$$

Equation (6) simply is the materials-balance-equation for one additional unit of resource input employed in the production process. It leaves the process either as part of the desired product or as waste. The amount of the latter, $\partial W/\partial R$, obviously, cannot be negative. Therefore, rearranging equation (6) into

$$\frac{\partial F(K,L,R)}{\partial R} = \frac{1}{\rho} \left(1 - \mu \, \frac{\partial W(K,L,R)}{\partial R} \right)$$

and noting that $\mu \partial W/\partial R \geq 0$ yields the following upper bound for the marginal resource product $\partial F/\partial R$:

$$\frac{\partial F(K,L,R)}{\partial R} \leq \frac{1}{\rho}. \tag{7}$$

This establishes the following result.

Proposition 2: *The marginal product of resource input, $\partial F/\partial R$, is bounded from above by the inverse of the resource fraction in the good produced, $1/\rho$.*

It is immediately obvious that if the marginal resource product F_R is bounded from above by $1/\rho$, then the marginal resource product as resource input approaches zero is also bounded from above be the same value:

$$\lim_{R \to 0} \frac{\partial F(K,L,R)}{\partial R} \leq \frac{1}{\rho} \; . \tag{8}$$

This is the content of the following corollary to proposition 2.

Corollary: *The marginal product of resource input as resource input approaches zero is bounded from above by the inverse of the resource fraction in the good produced,* $1/\rho$.

By this corollary it becomes apparent that the Inada conditions (in whatever form), when applied to a micro level production function with material resource inputs, are inconsistent with the Materials-Balance-Principle.

The intuition behind the simple formal exercise carried out in this section is that matter cannot be created and, consequently, the produced output cannot contain more of some material than has been supplied as input. If, for instance, one needs 100 gram of some resource material in order to produce 1 kilogram of a good ($\rho = 1/10$), then, out of 1 kilogram of the resource one can produce at maximum (i.e. with no waste) 10 kilogram of output. This means, the average as well as the marginal resource product is bounded from above by 10 ($= 1/\rho$).

4 Thermodynamic limits to resource productivity at the macro level

The simple model of production specified by equation (3) was confined to the description of one particular *micro level* production process and one particular natural resource. In order to analyse how the thermodynamic law of conservation of mass may restrict *aggregate* production, we should think of production in a more general way:

- There are many different natural resources, such that one can substitute from one resource to another, in order to avoid thermodynamic constraints on micro level production, such as conditions (5) or (7), to become binding.
- Production of an aggregate output, such as an all purpose commodity or GDP, is a multi-level process. On a first level, a number of different intermediate goods are produced from elementary resources (*micro level production*).

On a second level, the final output is produced from the intermediate goods (*macro level production*).[6]

- In the production of final output there is scope for substitution between the input of different intermediate goods.

In such a setting, there is plenty of scope for substitution both between different elementary resource materials and between production processes. The question then is: To what extent do thermodynamic constraints on micro level production processes, such as conditions (5) or (7), carry over to the macro level? And how do the laws of thermodynamics restrict aggregate production in such a general setting?

To answer these questions, consider the following model of production of an all purpose commodity from intermediate goods, which are themselves produced from a variety of elementary natural resources. There are n different elementary natural resources, numbered by $i = 1, \ldots, n$. Assume that this is a complete and exhaustive list of material resources actually or potentially used in production. For example, one may think of this list of natural resources as the complete list of known chemical elements, in which case $n = 112$.[7] These are used as inputs in the production of m different intermediate goods, each of which is produced by a single process of production, numbered by $j = 1, \ldots, m$. Production of theses intermediates is described by production functions

$$Y_j = F^j(K_j, L_j, R_{1j}, \ldots, R_{nj}) \qquad \text{for all } j = 1, \ldots, m, \tag{9}$$

where K_j and L_j denote input of capital and labor into production of intermediate good j. Similarly, R_{ij} (with $i = 1, \ldots, n$ and $j = 1, \ldots, m$) denotes input of

[6]In general, aggregate production may be over more than two levels. But the essential insights can already be grasped from considering a two-level-system.

[7]As of 2003, there are 112 known chemical elements, 83 of which are naturally occurring. Examples include hydrogen, carbon, oxygen, iron, copper, aluminum, gold and uranium. Elements 113 through 118 are known to exist, but are not yet discovered (IUPAC 2003).

resource material i into production of intermediate good j. Then,

$$R_i = \sum_{j=1}^{m} R_{ij} \tag{10}$$

is the total amount of resource material i utilized in production. Each production process F^j also yields a certain amount of waste, W_j.

Let ρ_{ij} with $0 \leq \rho_{ij} \leq 1$ denote the (mass) fraction of resource material i contained in the intermediate good j, and μ_{ij} with $0 \leq \mu_{ij} \leq 1$ the (mass) fraction of resource material i contained in the waste from producing intermediate good j. Note that ρ_{ij} will be zero if the intermediate good j does not contain any material of type i. This may include cases in which some resource material has been used in, or is even essential for, the production of the intermediate, say as a catalyst, yet the material is not contained in the good produced. Nonetheless, every intermediate good j – as long as it is a material good and not an immaterial service – contains *some* amount of *some* of the materials, while not containing anything of other materials. In order to make this distinction explicit, let

$$I_j = \{i \,|\, \rho_{ij} > 0\} \subseteq \{1, \ldots, n\} \tag{11}$$

be the set of all resources which make up – as far as material content goes – the intermediate good j. The complement set $\bar{I}^j = \{1, \ldots, n\} \setminus I^j$ then denotes the set of all resources which are not materially contained in the intermediate good j.

The final good, an all purpose commodity, is produced from capital K, labor L and the intermediate goods Y_j (with $j = 1, \ldots, m$):

$$Y = F(K, L, Y_1, \ldots, Y_m), \tag{12}$$

where Y_j (with $j = 1, \ldots, m$) denotes input of intermediate good j as produced on the first level of production (equation 9). On this level, elementary resources do not enter directly into production, but only indirectly insofar as they are embedded in the intermediates.[8] The final good production function (12) can be interpreted as an aggregate production function of the economy, specifying how the final good is

[8] This assumption, which is also quite plausible, only serves to simplify the notation and does not restrict the validity of results. It could easily be relaxed.

produced from elementary resources, when the Y_j's are replaced by the respective micro level production functions (equations 9).

The production of the final good also yields a certain amount of waste, W. Let ρ_i with $0 \leq \rho_i \leq 1$ denote the (mass) fraction of resource material i contained in the final output, and μ_i with $0 \leq \mu_i \leq 1$ the (mass) fraction of the resource material i contained in the waste. Note that ρ_i will be zero if the final good does not contain any material of type i. Nevertheless, the final good – as long as it is a material good and not an immaterial service – contains *some* amount of *some* of the materials, while not containing anything of other materials. For example, a passenger car may contain aluminum, carbon and thallium, but no gold or plutonium. In order to make this distinction explicit, let

$$I = \{i \,|\, \rho_i > 0\} \subseteq \{1, \ldots, n\} \tag{13}$$

be the set of all resources which make up – as far as material content goes – the final good. The complement set $\bar{I} = \{1, \ldots, n\} \setminus I$ then denotes the set of all resources which are not materially contained in the final good. Assume that the final good is material, that is, it contains at least one type of material.

ASSUMPTION 1: *I is non-empty.*

With this setting and notation, propositions 1 and 2 derived in section 3 above can obviously be translated and generalized into the following statement:

LEMMA 1: *The thermodynamic law of conservation of mass implies that the micro level production functions $F^j(K_j, L_j, R_{1j}, \ldots, R_{nj})$ for all $j = 1, \ldots, m$ have the following properties:*

$$\begin{aligned} \frac{F^j(K_j, L_j, R_{1j}, \ldots, R_{nj})}{R_{ij}} &\leq \frac{1}{\rho_{ij}} \qquad \textit{and} \\ \frac{\partial F^j(K_j, L_j, R_{1j}, \ldots, R_{nj})}{\partial R_{ij}} &\leq \frac{1}{\rho_{ij}} \qquad \textit{for all } i \in I_j. \end{aligned}$$

In words, the average and marginal resource product of resource material i in producing the intermediate good j is bounded from above by $1/\rho_{ij}$ in all cases where material i is contained in intermediate good j. If, in contrast, material i is not contained in intermediate good j, the average and marginal resource product of resource material i do not need to be bounded from above.[9]

Considering the overall two-level production system, the formal balance conditions for resource material i (for all $i = 1, \ldots, n$) then read as follows:

$$
\begin{aligned}
R_i &= \sum_{j=1}^{m} R_{ij} \ , && (14)\\
R_{ij} &= \rho_{ij} F^j(K_j, L_j, R_{1j}, \ldots, R_{nj}) + \mu_{ij} W_j \\
&\quad \text{for all } j = 1, \ldots, m \ , && (15)\\
\sum_{j=1}^{m} \rho_{ij} F^j(K_j, L_j, R_{1j}, \ldots, R_{nj}) &= \rho_i F(K, L, Y_1, \ldots, Y_m) + \mu_i W \ . && (16)
\end{aligned}
$$

Equation (14) states that the total amount of resource material i employed in production, R_i, may be used in any of the m production processes for intermediate goods. Equation (15) expresses conservation of mass of resource material i on the first level of production in all of the m intermediate good production processes: the total amount of material utilized in one of these processes, R_{ij}, leaves the process either as part of the intermediate good j or as part of the waste generated by that process. Equation (16) expresses conservation of mass on the second level of production: resource material i enters production of the final product indirectly, namely embedded in the m intermediate goods, each of which has a material content of that material of $\rho_{ij} Y_j$. It leaves the production process either as part of the final good or as part of the waste generated by that process. Summing balance equation (15) over all micro level processes j, adding balance equation (16) for the macro level, and using (14) yields an overall balance condition for material i:

$$
R_i = \rho_i F(K, L, Y_1, \ldots, Y_m) + \ \mu_i W + \sum_{j=1}^{m} \mu_{ij} W_j \ . \qquad (17)
$$

This condition states that the material utilized in production, R_i, leaves the two-level production system either as part of the final good, or as part of the waste generated by the final good production process, or as part of the waste generated in any of the m intermediate good production processes.

[9]Of course, in the latter case they may still be bounded from above for reasons other than thermodynamic necessity.

Rearranging equation (17) into

$$\frac{F(K, L, Y_1, \ldots, Y_m)}{R_i} = \frac{1}{\rho_i} \left[1 - \frac{\mu_i W}{R_i} - \frac{\sum\limits_{j=1}^{m} \mu_{ij} W_j}{R_i} \right] \tag{18}$$

and noting that $\mu_i W / R_i \geq 0$ as well as $\sum_{j=1}^{m} \mu_{ij} W_j / R_j \geq 0$, the following inequality holds:

$$\frac{F(K, L, Y_1, \ldots, Y_m)}{R_i} \leq \frac{1}{\rho_i}. \tag{19}$$

For all materials $i \in I$ which make up the final good, ρ_i is strictly positive so that $1/\rho_i < +\infty$ is a a finite upper bound for the average resource product of material i in aggregate production, F/R_i. From assumption 1 it follows that there is at least one such material. For all other materials with $i \notin I$, ρ_i is zero so that $1/\rho_i$ is not a finite upper bound. This establishes the following result.

Proposition 3:

(i) For all materials $i \in I$, which make up the final good, the average product of resource material i in aggregate production, F/R_i, is bounded from above by the inverse of this material's mass fraction in the final good, $1/\rho_i$.

(ii) There exists at least one such material for which the average product is bounded from above.

(iii) For all materials $i \notin I$, which are not contained in the final good, the average product of resource material i in aggregate production, F/R_i, does not need to be bounded from above.

In order to derive an analogue statement about the marginal resource products, take the total differential of the material balance equation (17) for material i, with the Y_j in production function F replaced my the intermediate good production functions F^j (equations 9), and consider only variations in resource material i (i.e. $dK = dL = 0$, $dK_j = dL_j = 0$ for all j and $dR_{i'j} = 0$ for all $i' \neq i$):

$$dR_i = \rho_i \sum_{j=1}^{m} \frac{\partial F}{\partial Y_j} \frac{\partial F^j}{\partial R_{ij}} \, dR_{ij} + \ \mu_i \sum_{j=1}^{m} \frac{\partial W}{\partial Y_j} \frac{\partial F^j}{\partial R_{ij}} \, dR_{ij} + \sum_{j=1}^{m} \mu_{ij} \frac{\partial W_j}{\partial R_{ij}} \, dR_{ij} \ . \tag{20}$$

From balance equation (14) it follows that

$$dR_i = \sum_{j=1}^{m} dR_{ij} \ . \tag{21}$$

Replacing dR_i in equation (20) by expression (21) and rearranging terms yields

$$\sum_{j=1}^{m} \left[1 - \rho_i \frac{\partial F}{\partial Y_j} \frac{\partial F^j}{\partial R_{ij}} - \mu_i \frac{\partial W}{\partial Y_j} \frac{\partial F^j}{\partial R_{ij}} - \mu_{ij} \frac{\partial W_j}{\partial R_{ij}} \right] dR_{ij} = 0 \ . \tag{22}$$

This holds for all $dR_{ij} \geq 0$ and, thus, implies that the term in brackets equals zero. This can be rearranged into

$$\frac{\partial F}{\partial Y_j} \frac{\partial F^j}{\partial R_{ij}} = \frac{1}{\rho_i} \left[1 - \mu_i \frac{\partial W}{\partial Y_j} \frac{\partial F^j}{\partial R_{ij}} - \mu_{ij} \frac{\partial W_j}{\partial R_{ij}} \right] \ . \tag{23}$$

Noting that the second and third term in brackets are non-negative yields the following inequality, which holds for all i and j:

$$\frac{\partial F}{\partial Y_j} \frac{\partial F^j}{\partial R_{ij}} \leq \frac{1}{\rho_i} \ . \tag{24}$$

On the other hand, taking the total differential of the defining equation for production function F (equation 12), with the Y_j in production function F replaced my the intermediate good production functions F^j (equations 9), and considering only variations in resource material i (i.e. $dK = dL = 0$, $dK_j = dL_j = 0$ for all j and $dR_{i'j} = 0$ for all $i' \neq i$), yields:

$$dF = \sum_{j=1}^{m} \frac{\partial F}{\partial Y_j} \frac{\partial F^j}{\partial R_{ij}} dR_{ij} \ . \tag{25}$$

From (25) one obtains, using inequality (24) and equation (21)

$$dF = \sum_{j=1}^{m} \frac{\partial F}{\partial Y_j} \frac{\partial F^j}{\partial R_{ij}} dR_{ij} \leq \sum_{j=1}^{m} \frac{1}{\rho_i} dR_{ij} = \frac{1}{\rho_i} \sum_{j=1}^{m} dR_{ij} = \frac{1}{\rho_i} dR_i \ , \tag{26}$$

so that we have the following inequality:

$$dF \leq \frac{1}{\rho_i} dR_i \ . \tag{27}$$

Interpreting this inequality for differentials as an algebraic expression and rearranging finally yields:

$$\frac{dF}{dR_i} \leq \frac{1}{\rho_i}. \tag{28}$$

Since the production function F (equation 12) does neither directly nor indirectly depend on R_i, the expression dF/dR_i should not be interpreted as a (total) derivative in the strict sense. However, in a rather loose sense, it may be interpreted as something like a total derivative. It tells us by how much the aggregate output Y changes when an additional marginal unit of resource material R_i is used in production, by dividing it up among the m intermediate good production processes in such a manner that $dR_i = \sum_{j=1}^{m} dR_{ij}$.[10]

Again, for all materials $i \in I$ which make up the final good, ρ_i is strictly positive so that, according to inequality (28), $1/\rho_i < +\infty$ is a a finite upper bound on dF/dR_i. From assumption 1 it follows, that there is at least one such material. For all other materials with $i \notin I$, ρ_i is zero so that $1/\rho_i$ is not a finite upper bound. This establishes the following result.

Proposition 4:

(i) For all materials $i \in I$, which make up the final good, the marginal product of resource material i in aggregate production, dF/dR_i, is bounded from above by the inverse of this material's mass fraction in the final good, $1/\rho_i$.

(ii) There exists at least one such material for which the marginal product is bounded from above.

(iii) For all materials $i \notin I$, which are not contained in the final good, the marginal product of resource material i in aggregate production, dF/dR_i, does not need to be bounded from above.

Comparing propositions 3 and 4 for macro level production with propositions 1 and 2 for micro level production, we see that all results that were obtained in the

[10] In that sense, one could define

$$\frac{dF}{dR_i} \equiv \sum_{j=1}^{m} \frac{\partial F}{\partial Y_j} \frac{\partial F^j}{\partial R_{ij}} \quad \text{subject to} \quad dR_i = \sum_{j=1}^{m} dR_{ij} \; .$$

However, this definition is not unique. While there is a multitude of ways in which dR_i may be divided up among the m different intermediate good production processes, inequality (27) holds in any case. Hence, result (28) holds irrespective of the exact way in which dF/dR_i is defined.

simple micro-level setting essentially carry over to the general two-level-multi-resources-multi-processes setting. The only qualification is that the boundedness-results only hold for materials in the set I which make up the final good.

5 Discussion

It has been shown that the Inada conditions, when applied to material resource inputs, may be inconsistent with the thermodynamic law of conservation of mass, the so-called Materials-Balance-Principle. In particular, the analysis has revealed that the average and marginal product of a natural resource material in aggregate production are bounded from above due to the thermodynamic law of mass conservation if the final good, an all-purpose commodity, contains this material. An upper bound is given by the inverse of this material's mass fraction in the final good.

The analysis was based on a model of multi-level production where different intermediate goods are produced from different elementary resources, and an all-purpose final commodity is produced from these intermediates. Note that no limits on substitution between resource materials or between intermediate products have been assumed. Another thing to note is that the upper bound specified by inequalities (19) and (28) is certainly not the lowest upper bound, but comes out of a more or less crude estimation (from equations (18), (26) to inequalities (19), (27)). For that reason, the upper bound given here does not depend on any model parameters other than ρ_i.

When discussing the relevance of these results for the natural-resources-and-economic-growth-debate, the crucial questions are:

(i) How many, and which natural resource materials are elements of the set I? That is, what are the natural resource materials that make up, materially, the final good?

(ii) What is these materials' mass fraction ρ_i in the final good?

(iii) How do the set I and the relevant parameter values ρ_i change over time?

It is probably the difference in opinion on these questions which make a difference between the "optimists" and the "pessimists" in the discussion about the thermodynamic limits to economic growth.

This analysis has revealed that there are stringent thermodynamic limits to resource productivity in aggregate production for a number of natural resource materials. The analysis has also revealed that not all resource materials need to be limiting. Hence, the overall conclusion is that the question of thermodynamic limits to economic growth requires a detailed investigation, with separate analyses and results for each material. This shifts the focus of the debate from overall growth to the more detailed level of factor substitution and structural change.

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