Unambiguous Events and Dynamic Choquet Preferences

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Abstract

This paper explores the relationship between dynamic consistency and existing notions of unambiguous events for Choquet expected utility preferences. A decision maker is faced with an information structure represented by a filtration. We show that the decision maker's preferences respect dynamic consistency on a fixed filtration if and only if the last stage of the filtration is composed of unambiguous events in the sense of Nehring (1999). Adopting two axioms, conditional certainty equivalence consistency and constrained dynamic consistency to filtration measurable acts, it is shown that the decision maker respects these two axioms on a fixed filtration if and only if the last stage of the filtration is made up of unambiguous events in the sense of Zhang (2002).

Keywords: Choquet expected utility, unambiguous events, filtration, updating, dynamic consistency, consequentialism

JEL-Codes: D81, D82

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1 Introduction

The objective of this paper is to explore the link between updating nonexpected utility preferences and two existing notions of unambiguous events. Nonexpected utility preferences are widely used in modeling ambiguity averse behavior, as exemplified by the famous Ellsberg (1961) paradox. Choquet expected utility (CEU) theory is a prominent class of such preferences. As a decision criterion it was axiomatically justified, in different frameworks, by Schmeidler (1989), Gilboa (1987) and Sarin and Wakker (1992). According to this theory, decision makers' beliefs are represented by *capacities*, which are not-necessarily-additive probabilities. To make nonexpected utility models attractive for economic and game theoretic applications it is important to know how well do they perform in dynamic choice situations. In this contribution we ask whether for CEU preferences the property of dynamic consistency, constrained to a given collection of events, guarantees that its elements are unambiguous and vice versa. The results we obtain allow us to answer this question in the affirmative.

Recently, several extensions of CEU preferences to intertemporal decision making have been proposed (see e.g. Machina (1989), Sarin and Wakker (1998a), Eichberger, Grant, and Kelsey (2005), Eichberger, Grant, and Kelsey (2007)). In dynamic choice situations the decision maker (henceforth DM) receives new information, updates preferences, and formulates a new plan of action. A fundamental question raised in this context is how updated preferences that govern future choices are linked to contingent choices made ex ante. Two axioms are used to justify the link: *dynamic consistency* and *consequentialism*. Dynamic consistency requires that choices made ex ante are respected by updated preferences. Consequentialism requires that after being informed that some event occurred, the conditional preferences are not affected by outcomes outside of that event. When the DM's preferences satisfy both axioms simultaneously on all events, then these preferences admit the expected utility representation and the updating rule coincides with the Bayes revision rule. This result was obtained in the presence of risk by Hammond (1988) and in the presence of uncertainty by Ghirardato (2002) and belongs today to the 'folk wisdom' of the decision theory. However, as the following dynamic version of the classical 3-color Ellsberg experiment illustrates, it is impossible to retain both rationality arguments on all events in the presence of ambiguity. As a mind experiment it was described by Ghirardato, Maccheroni, and Marinacci (2008) and Siniscalchi (2009). Recently, Dominiak, Dürsch, and Lefort (2009) ran a real experiment on this issue.

Example 1.1. *The DM maker faces an urn containing 90 balls, 30 of which are known to be red* $\{R\}$ *and 60 of which are somehow divided between blue* $\{B\}$ *and yellow* $\{Y\}$*, with no further information on the distribution. At the ex ante stage* ($t = 0$) *the DM only has the information as described above. Suppose that at the interim stage* ($t = 1$) *one ball is drawn from the urn at random, and then the DM is informed that the ball is not yellow, i.e.* $\{R, B\}$*. The DM has to choose between two bets* (f, f') *and* (g, g') *paying off* 100 *or* 0*, depending on the color of the randomly drawn ball. Suppose that at the ex ante stage* ($t = 0$) *the DM, like a majority of subjects in experimental studies (see Camerer and Weber (1992)), displays the following patterns of preferences:*

$$f = \begin{pmatrix} 100 & \text{if } \omega \in R \\ 0 & \text{if } \omega \in B \\ 0 & \text{if } \omega \in Y \end{pmatrix} \succ \begin{pmatrix} 0 & \text{if } \omega \in R \\ 100 & \text{if } \omega \in B \\ 0 & \text{if } \omega \in Y \end{pmatrix} = f'$$

$$g = \begin{pmatrix} 100 & \text{if } \omega \in R \\ 0 & \text{if } \omega \in B \\ 100 & \text{if } \omega \in Y \end{pmatrix} \prec \begin{pmatrix} 0 & \text{if } \omega \in R \\ 100 & \text{if } \omega \in B \\ 100 & \text{if } \omega \in Y \end{pmatrix} = g'$$

A DM displaying such preferences is reluctant to bet on events with unknown probabilities and therefore she is said to be averse toward ambiguity. Now consider the interim stage ($t = 1$) *and the possible patterns of conditional preferences,* $(\succeq_{\{R,B\}})$*, which respect dynamic consistency,* (i)*, and which respect consequentialism,* (ii)*.*

(i) *According to the property of dynamic consistency the DM's conditional preferences have to respect the choices made ex ante, i.e.* $f \succ_{\{R,B\}} f'$ *and* $g \prec_{\{R,B\}} g'$*.*

(*ii*) *According to consequentialism, since the bets f, g and f', g' are the same on the event $\{R, B\}$ and differ only outside that event, the DM must be conditionally indifferent between them, i.e. $f \sim_{\{R,B\}} g$ and $f' \sim_{\{R,B\}} g'$. Furthermore, consequentialism implies that if $f \succ_{\{R,B\}} f'$, then $g \succ_{\{R,B\}} g'$ and vice versa (respectively if $f \prec_{\{R,B\}} f'$, then $g \prec_{\{R,B\}} g'$ and vice versa).*

It can be immediately seen that in dynamic situations an ambiguity averse DM must violate either the property of dynamic consistency or consequentialism (or both). Then, if conditional preferences respect dynamic consistency, as in (i), the property of consequentialism is violated. On the other hand, if the conditional preferences remain consistent with consequentialism, as in (ii), exactly one of the ex ante preferences is reversed, what violates dynamic consistency.[1]

As an immediate consequence, when extending nonexpected utility models to dynamic frameworks, we must either relax the property of dynamic consistency or consequentialism, or we may maintain both rationality arguments, but, constrain the analysis to some fixed collection of events. We follow the latter direction and characterize the properties of events on which dynamic consistency and consequentialism are satisfied. A natural candidate for events on which both axioms are satisfied, are events that support some kind of probabilistic beliefs, as for instance events with known distribution, i.e. $\{R\}$ and $\{B, Y\}$ in the example above. The idea of events characterized by probabilistic beliefs is closely related to the recently suggested notions of *unambiguous* events by Nehring (1999), Epstein and Zhang (2001), Zhang (2002) and Ghirardato, Maccheroni, and Marinacci (2004).

First, we focus on the definition due to Nehring (1999), since it mimics the desirable separability property of expected utility theory.[2] His definition is based

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[1] If conditionally on the event $\{R, B\}$ the DM reverses both ex ante preferences, i.e. $f \prec_{\{R,B\}} f'$ and $g \succ_{\{R,B\}} g'$, then such DM is inconsistent with dynamic consistency and consequentialism.

[2] Separability of preferences and beliefs is a key property of expected utility theory. It means that subjective probabilities assigned to uncertain events are not affected by outcomes that are associated to these events. This property is also satisfied by the more general class of preferences, called probabilistically sophisticated preferences axiomatized by Machina and Schmeidler (1992).

on the idea, originated by Sarin and Wakker (1998b), to interpret capacities in terms of *rank dependent probability assignments*. According to this interpretation, subjective probabilities used for evaluating acts depend on the rank ordering of their consequences. In general two acts generating distinct ranks are evaluated with respect to different subjective probabilities. Thus, the separability of preferences and beliefs may be achieved for acts that generate the same rank. Such acts are called *comonotonic*. In the case that the subjective likelihood of an event is unaffected by changing its position, it must be viewed as unambiguous. Correspondingly, Nehring (1999) calls an event unambiguous, henceforth *N-unambiguous*, if the subjective probability attached to the event does not depend on the ranking position of states.

We argue that conditional on N-unambiguous events, the Bayes updating rule for capacities is the most appropriate updating rule. The reason is twofold. First, because updating on N-unambiguous events according to the Bayes revision rule is the only way to retain dynamic consistency. Second, when conditioning on N-unambiguous events, the Bayesian updating rule coincides with other popular updating rules. These include the Full-Bayesian updating rule introduced by Jaffray (1992) and all the h-Bayesian updating rules as axiomatized by Gilboa and Schmeidler (1993). Motivated by this rationale we show that consequentialist Choquet expected utility preferences satisfy dynamic consistency on a fixed filtration if and only if the algebra generated by the smallest elements in the filtration belongs to the algebra generated by N-unambiguous events. This result on its own may be viewed as an alternative characterization of N-unambiguous events in a conditional decision problem.

Furthermore, Nehring (1999) emphasized the restrictiveness of CEU preferences, since the collection of N-unambiguous events must be always an algebra. However, there may be potentially interesting ambiguity situations, as exemplified by Zhang (2002) in his 4-color example, in which the candidates for unambiguous events form a weaker structure. By departing from the intuition behind Savage's key axiom, called the Sure-Thing-Principle, Zhang (2002) suggested a weaker definition of unambiguous events, henceforth *Z-unambiguous*. Thus, it is impossible to maintain dynamic consistency on events that are Z-unambiguous. An illustrating dynamic extension of the 4-color example is given in Section 5. Adopting an axiom, called *conditional certainty equivalence consistency* and constraining the dynamic consistency to *partition measurable acts*, we provide a dynamic characterization of Z-unambiguous events in a conditional decision problem.

This paper is organized as follows. Section 2 presents the necessary notation. In Section 3 the definitions of N-unambiguous events and Z-unambiguous events are introduced. Section 4 presents the main concepts regarding the conditional decision problem. In Section 5 we provide a characterization of N-unambiguous events in a conditional decision problem. Moreover, we make some remarks on the related literature. In Section 6 we provide an illustrative example and establish a dynamic characterization of Z-unambiguous events. Finally, we conclude in Section 7.

2 Notation

The uncertainty, which the DM faces, is described by a finite set of states of nature, Ω. An event A is a subset of Ω. The algebra generated by Ω is denoted by $\mathcal{A}$. For all $A \subset \Omega$, we denote $\Omega \setminus A$, the complement of A, by A^c. Let X be the set of outcomes. An act f is a function from Ω to X. For instance, an act $f = (A_1, x_1; \ldots; A_n, x_n)$ assigns the outcome x_j to each $\omega \in A_j$, $j = 1, \ldots, n$, where $A_1, \ldots, A_n$ are events partitioning Ω. Let $f_A g$ be an act that assigns the outcome $f(\omega)$ to each $\omega \in A$ and the outcome $g(\omega)$ to each $\omega \in A^c$. An act $f = x$ that assigns a constant outcome to each $\omega \in \Omega$ is called a constant act. Denote the set of all acts by $\mathcal{F}$. A set function $\nu : \mathcal{A} \to \mathbb{R}$ is called *capacity* if it satisfies the following conditions: (1) $\nu(\emptyset) = 0$ and $\nu(\Omega) = 1$; (2) if $A \subset B \subset \Omega$, then $\nu(A) \leq \nu(B)$. Let $\succeq$ be a binary relation on the set of acts, $\mathcal{F}$, that represents preferences. The DM is said to have Choquet expected utility preferences, if there exists a utility function $u : X \to \mathbb{R}$ and a capacity ν such that, for all $f, g \in \mathcal{F}$, $f \succeq g$ if and only if $\int_\Omega u \circ f d\nu \geq \int_\Omega u \circ g d\nu$. Formally, expected utility of an act f with respect to the utility index u and the capacity ν is defined as:

$$\int_\Omega u \circ f \; d\nu = u(f(A_1)) + \sum_{i=2}^{n} \left[u(f(A_i)) - u(f(A_{i-1}))\right] \nu(A_i, \ldots, A_n)$$

with $\{A_i\}_{i=1,\ldots,n}$ chosen such that $u(f(A_1)) \leq u(f(A_2)) \leq \cdots \leq u(f(A_n))$. Schmeidler (1989), Gilboa (1987) and Sarin and Wakker (1992) axiomatized CEU preferences for an infinite state space. Assuming a rich set of consequences and allowing for a finite state space CEU preferences were axiomatized by Wakker (1989), Nakamura (1990) and Chew and Karni (1994).

Throughout the paper we assume that preferences are represented by CEU. Additionally we restrict the set of outcomes X and preferences $\succeq$ on $\mathcal{F}$ by assuming that:

Assumption 1. (Continuity) The utility function $u : X \to \mathbb{R}$ is continuous.

Assumption 2. (Solvability) For any $f \in \mathcal{F}$ there exists $x \in X$ such that $f \sim x$.

Solvability serves as a richness condition on $\succeq$ and X. It is satisfied in all axiomatizations of CEU in finite state space set-up. For instance, Nakamura (1990) and Chew and Karni (1994) impose it directly on $\succeq$, while Wakker (1989) requires X to be a connected and separable topological space.

3 Unambiguous events

This section provides a behavioral characterization of unambiguous events. We begin with the characterization suggested by Nehring (1999), who interprets capacities in terms of *rank dependent probability assignments.* Let ρ be a bijection $\rho : \Omega \to \{n, \ldots, 1\}$. The mapping ρ expresses the *ranking* position of states associated with an act f, i.e. the favorableness of their outcome relative to the outcomes obtained under other states. Let $\mathcal{R}$ be a set of such rankings and let Δ^Ω be a set of probability distributions over Ω. Two ranks ρ and ρ', for which at most two adjacent states swapped their ranking position, are said to be neighboring ranks. Formally, we say that ρ is a neighbor of ranking ρ', written $\rho N \rho'$, if and only if for at most two states

$\omega \in \Omega$, $\rho(\omega) = \rho'(\omega)$, and for all $\omega \in \Omega$, $|\rho(\omega) - \rho'(\omega)| \leq 1$. A mapping $m : \mathcal{R} \to \Delta^{\Omega}$ is called *rank dependent probability assignment* if and only if for all $\rho, \rho' \in \mathcal{R}$ such that $\rho N \rho'$, and all $\omega \in \Omega$ such that $\rho(\omega) = \rho'(\omega)$: $m_\rho(\omega) = m_{\rho'}(\omega)$. For a given capacity ν on Ω the rank dependent probability assignment m_ρ may be defined as follows $m_\rho(\omega) = \nu(\omega' : \rho(\omega') \leq \rho(\omega)) - \nu(\omega' : \rho(\omega') < \rho(\omega))$.[3] The mapping m_ρ may be interpreted as the marginal capacity contribution of the state ω to all states yielding better outcomes. The Choquet integral of an act f with respect to ν and u can be written as the Choquet integral with respect to m_ρ and u, i.e.:

$$\int_\Omega u \circ f \, d\nu = \int_\Omega u \circ f \, dm_\rho = u(f(A_1)) + \sum_{i=2}^{n} \left[u(f(A_i)) - u(f(A_{i-1})) \right] m_\rho(A_i, \ldots, A_n).$$

By abuse of notation, we denote a measure $m_{\rho(f)}$, such that $m_{\rho(f)}(A_i, \ldots, A_n) = v(A_i, \ldots, A_n)$ with $1 \leq i \leq n$, as the rank dependent probability assignment m_ρ associated with an act f. Thus, throughout the paper we write the Choquet expectation of f, taken with respect the measure $m_{\rho(f)}$, as $\int_\Omega u \circ f \, d\nu = \int_\Omega u \circ f \, dm_{\rho(f)}$. Call a pair of acts f and g *comonotonic*, if there are no two states ω, ω' such that $f(\omega) < f(\omega')$ and $g(\omega) > g(\omega')$. For any act g, comonotonic with f and measurable with respect to f, the Choquet integral of g with respect to ν and u is equal to the expectation of g with respect to $m_{\rho(f)}$ and u.

According to this interpretation, an event A is called *N-unambiguous* if its rank dependent probability assignment does not depend on its ranking.

Definition 3.1. *Fix an event $A \in \mathcal{A}$. A is N-unambiguous if $m_\rho(A) = \nu(A)$ for all $\rho \in \mathcal{R}$, otherwise A is N-ambiguous.*

Let $\mathcal{A}_N^U$ be the set of all N-unambiguous events. Nehring (1999) proves that for any capacity ν the set $\mathcal{A}_N^U$ is an algebra. Moreover, any capacity ν is always additively separable across its unambiguous events. Thus, $\mathcal{A}_N^U = \{A | v(B) = v(B \cap A) + v(B \cap A^c)$ for all $B \in \mathcal{A}\}$. An alternative way to define N-unambiguous events is to use Savage's

[3] Nehring (1999) showed that there is a one-to-one relation between capacities and rank dependent probability assignments, m_ρ. In his definition the superscript ν is used for m_ρ^ν. We drop it for notational simplicity.

axiom called the Sure-Thing-Principle. However, since the Sure-Thing-Principle, applied to the whole algebra of events $\mathcal{A}$, implies that beliefs are probabilistic, we have to constrain its domain to some events. Thus, we say that the Sure-Thing-Principle holds at A and A^c if and only if for any act $f, f', g, g' \in \mathcal{F}$: if $f_D g \succeq f'_D g$, then $f_D g' \succeq f'_D g'$ and $D \in \{A, A^c\}$.

Proposition 3.1. *Fix an event $A \in \mathcal{A}$. The following two statements are equivalent:*

i) A is N-unambiguous, i.e. $A \in \mathcal{A}_N^U$.

ii) The Sure-Thing-Principle at A and A^c is satisfied.

Ghirardato, Maccheroni, and Marinacci (2004) provide the behavioral counterpart to N-unambiguous events in a different setup, assuming a convex structure on the set of consequences. In particular, an event A is N-unambiguous if for any $x, x' \in X$ bets of the form $x_A x'$ cannot not be used for hedging other acts. According to their proposition 10 all such bets (called crisp acts) are evaluated with respect to the same probability distribution. Thus, the measure of an event A, $m_{\rho(x_A x')}(A)$, is independent of the rank ρ, meaning that A is N-unambiguous event.

Zhang (2002) constrains Savage's Sure-Thing-Principle and proposes a weaker definition of unambiguous events. An event A is *Z-unambiguous* if replacing an outcome that was constant over A by any other constant outcome does not change the ranking of the pair of acts being compared. The following definition reflects this idea.

Definition 3.2. *An event $A \in \mathcal{A}$ is Z-unambiguous if for any $f, g \in \mathcal{F}, x \in X$*

$$f_A x \succeq g_A x \Rightarrow f_A x' \succeq g_A x'$$

for any $x' \in X$ and the same implication holds for A^c. Otherwise A is Z-ambiguous.

Let $\mathcal{A}_Z^U$ be the set of all Z-unambiguous events. It is well known that $\mathcal{A}_N^U \subset \mathcal{A}_Z^U$. Note that $\mathcal{A}_Z^U$ does not need to be an algebra. It is a λ-system, a collection of events closed under complements and disjoint unions, but not under intersections.[4] Furthermore, Zhang (2002) established the following characterization of $\mathcal{A}_Z^U$ in terms of capacities. An event A is Z-unambiguous if and only if $v(A \cup B) = v(A) + v(B)$ for all $B \subset A^c$ and $v(A^c \cup C) = v(A^c) + v(C)$ for all $C \subset A$.

4 Information and Updating

We limit our attention to updating on events that the DM views as possible to occur, i.e. non-null events. An event $A \in \mathcal{A}$ is non-null if $\nu(A) > 0$.[5] As time progresses the DM is informed that the true state of the nature ω is an element of an event A, i.e. $\omega \in A$. A natural way to model information is by means of event trees represented by a filtration. We assume that time is discrete, finite and goes over the index set $\mathcal{T} = \{0, \ldots, T\}$. Let $\mathcal{P}_t$ be a partition of the state space Ω. A filtration $\mathcal{P} = \{\mathcal{P}_t\}_{t \in \mathcal{T}}$ is a collection of partitions such that $\mathcal{P}_0 = \{\Omega\}$, any $\mathcal{P}_{t+1}$ is finer than $\mathcal{P}_t$ for all $t < T$, and $\mathcal{P}_T = \{\{\omega\} : \omega \in \Omega\}$. A filtration is given and fixed throughout. Let $\mathcal{A}_{\mathcal{P}}$ be the algebra generated by the smallest elements of a given filtration $\mathcal{P}$.

At the ex ante stage, $t = 0$, the DM formulates a complete contingent plan of action. When no information is given, the relation $\succeq$ represents the DM's unconditional preferences, that is $\succeq$ is equivalent to $\succeq_\Omega$. At any interim stage, $t < T$, the DM faces new information and has a chance to review the contingent plan for the remaining time periods. We denote by $\succeq_A$ the CEU preferences over $\mathcal{F}$ conditional on $A \in \mathcal{P}_t$, i.e. for all $f, g \in \mathcal{F}$,

$$f \succeq_A g \Leftrightarrow \int_\Omega u \circ f d\nu_A \geq \int_\Omega u \circ g d\nu_A$$

with ν_A a capacity conditional on A. As in Ghirardato (2002) we reduce conditional decision problems to static ones.

[4]For more general preferences than Choquet preferences Kopylov (2007) showed that $\mathcal{A}_Z^U$ is weaker than originally claimed λ-systems, it is a mosaic. A mosaic is a collection of events closed under complements but not under all disjoint unions.

[5]When an event A is either N-unambiguous or Z-unambiguous this definition of null events is equivalent to the stronger notion, A is null if $\nu(A \cup B) = \nu(B)$ for any $B \in \mathcal{A}$.

In a dynamic framework it is important to know how the conditional and the unconditional preferences are related to each other. The following three axioms impose dynamic restrictions on preferences over $\mathcal{F}$. The first property, called *consequentialism*, concerns only the conditional preference relation. It requires that preferences conditional on a non-null event A are not affected by outcomes outside the conditional event, A^c. Intuitively, once the DM is informed that an event A occurred, only the uncertainty about all subevents of A matters for preferences. The uncertainty about counterfactual events, A^c, is not relevant anymore for future choices.

Axiom 4.1 (Consequentialism). *For any non-null $A \in \mathcal{A}$ and all $f, g \in \mathcal{F}$, $f(\omega) = g(\omega)$ for each $\omega \in A$ implies $f \sim_A g$.*

Throughout the paper, we assume that preferences satisfy consequentialism. An important axiom linking directly conditional and unconditional preferences is called *dynamic consistency.* Dynamic consistency requires that ex ante contingent choices are respected by updated preferences and vice versa.

Axiom 4.2 (Dynamic consistency). *For any non-null $A \in \mathcal{A}$ and all $f, g \in \mathcal{F}$ such that $f(\omega) = g(\omega)$ for each $\omega \in A^c$, $f \succeq g \Leftrightarrow f \succeq_A g$.*

Essentially, when the DM prefers f to g without any information regarding A, and f and g are the same outside of A, she should also prefer f to g after being informed that A occurred and vice versa.

The third property, called *conditional certainty equivalent consistency*, is adapted from Pires (2002).[6] This property is a weaker version of dynamic consistency. It states: if and only if conditional on a non-null event A, the DM is indifferent between the act f and the constant payment x, then the unconditional preferences should also express indifference between the outcome x and the act $f_A x$, which agrees with the act f on A and otherwise assigns the constant outcome x.

Axiom 4.3 (Conditional certainty equivalent consistency). *For any non-null $A \in \mathcal{A}$ any outcome $x \in X$ and any $f \in \mathcal{F}$, $f \sim_A x \Leftrightarrow f_A x \sim x$.*

[6] In her paper Pires (2002) axiomatizes the Full-Bayesian updating rule for the multiple prior preferences of Gilboa and Schmeidler (1989).

At the interim stage, the revealed information is taken into account by updating the DM's subjective beliefs. For CEU preferences, there are several ways of defining the conditional capacity ν_A. The most common updating rules used to revise capacities are: the Bayes updating rule, the Dempster-Shafer updating rule and the Full-Bayesian updating rule. For the sake of completeness, we recall the respective definitions.

Definition 4.1. *Let ν be a capacity on Ω and let $A \subset \Omega$. If A is observed and $B \subset A$, then:*

i) the Bayes updating rule (B) is given by

$$\nu_A(B) = \frac{\nu(B \cap A)}{\nu(A)},$$

ii) the Dempster-Shafer updating rule (DS) of ν is given by

$$\nu_A^{DS}(B) = \frac{\nu((B \cap A) \cup A^c) - \nu(A^c)}{1 - \nu(A^c)},$$

ii) the Full-Bayesian updating rule (FB) is given by

$$\nu_A^{FB}(B) = \frac{\nu(B)}{1 - \nu(B \cup A^c) + \nu(B \cap A)}.$$

Gilboa and Schmeidler (1993) characterize behaviorally the Dempster-Shafer updating rule, introduced by Dempster (1968) and Shafer (1976). Eichberger, Grant, and Kelsey (2007) provide an axiomatic characterization of the Full-Bayesian updating rule for CEU preferences. Moreover, the Dempster-Shafer and the Bayes updating rule belong to the class of so called *h-Bayesian* updating rules introduced by Gilboa and Schmeidler (1993).

Definition 4.2 (h-Bayesian updating rule)**.** *There is an act $h \in \mathcal{F}$ such that for all $f, g \in \mathcal{F}$ and all $A \in \mathcal{A}$, $f \succeq_A g \Leftrightarrow f_A h \succeq g_A h$.*

When preferences admit a CEU representation then for the Dempster-Shafer (or *pessimistic*) updating rule, the act $h = x^*$ is a constant act yielding the most preferred outcome in X. That is, under the Dempster-Shafer updating rule, the conditionally null event, A^c, is associated with the best outcome possible. For the Bayes (or *optimistic*) updating rule the act $h = x_*$ is a constant act associating the worst possible outcome in X (note that w.l.o.g. we suppose that such x^* and x_* exist). According to Gilboa and Schmeidler (1993) the DM exhibits 'happiness' that an event A occurred and decisions are made as if she were always in 'the best of all possible worlds' ('happiness' comes from the fact that the event A^c, which did not occur, was associated by the DM with the worst outcomes). All the h-Bayesian updating rules satisfy consequentialism but not necessarily dynamic consistency.

5 N-Unambiguous Events in a Conditional Decision Problem

The objective of this section is to establish the necessary and sufficient conditions for CEU preferences to be dynamically consistent on events in a fixed filtration. We begin by looking for an appropriate updating rule on the filtration $\mathcal{P}$ made up of N-unambiguous events, i.e. $\mathcal{A}_{\mathcal{P}} \subset \mathcal{A}_N^U$. It turns out that the Bayes revision rule for capacities is the only way to ensure dynamic consistency on the filtration $\mathcal{P}$, whose elements are N-unambiguous events. Moreover, when the conditional event is N-unambiguous, then the property of conditional certainty equivalent consistency implies that beliefs are revised according to the Bayes updating rule. These observations are summarized in the following proposition.

Proposition 5.1. *Let ν be a capacity on Ω and let $A \in \mathcal{A}_N^U$ be a N-unambiguous event, then the following three statements are equivalent:*

i) Conditional certainty equivalent consistency is satisfied.

ii) The capacity ν is updated according to Bayes updating rule.

iii) Dynamic consistency is satisfied.

Remark 5.1. *Ghirardato, Maccheroni, and Marinacci (2008) provide a similar result for a larger class of preferences than the class of CEU preferences, the invariant* *biseparable preferences. However, the properties that they obtain are not available for all acts but only for acts which are unambiguous (i.e. acts measurable with respect to the unambiguous partition).*

As next we state that the Full-Bayesian updating rule and all the h-Bayesian updating rules coincide with the Bayes revision rule when the conditional event A belongs to the algebra generated by N-unambiguous events, i.e. $A \in \mathcal{A}_N^U$.

Proposition 5.2. *Let ν be a capacity on Ω and let $A \in \mathcal{A}_N^U$ be a N-unambiguous event, then the Full-Bayesian updating rule and all the h-Bayesian updating rules coincide with the Bayes updating rule.*

Now we are ready to state our first theorem. It claims that CEU preferences satisfy dynamic consistency on events in a fixed filtration if and only if the algebra generated by the events from that filtration belongs to the algebra generated by N-unambiguous events. Intuitively, CEU preferences respect dynamic consistency on a fixed collection of events, which are not affected by ambiguity.

Theorem 5.1. *Let $\mathcal{P} = \{\mathcal{P}_t\}_{t\in\mathcal{T}}$ be a fixed filtration on Ω and let $\mathcal{A}_{\mathcal{P}}$ be an algebra generated by $\mathcal{P}$. If the DM has CEU preferences then the following conditions are equivalent:*

i) The DM is dynamically consistent with respect to $\mathcal{P}$.

ii) $\mathcal{A}_{\mathcal{P}}$ belongs to $\mathcal{A}_N^U$ and ν is updated according to the Bayes updating rule.

Some remarks regarding the theorem and the related literature are in order.

Remark 5.2. *Our result extends the theorem of Eichberger, Grant, and Kelsey (2005), which is true only for convex capacities, to all capacities. Then, for a capacity ν being convex, the additivity on $A \in \mathcal{A}$, i.e. $\nu(A^c) + \nu(A) = 1$, is equivalent to A being N-unambiguous. The proof relies on their lemma 2.1 stating that if $\nu(A^c) + \nu(A) = 1$, then for any $B \in \mathcal{A}$, $\nu(B) = \nu(A^c \cap B) + \nu(A \cap B)$. Instead of assuming the Bayesian updating rule as in Eichberger, Grant, and Kelsey (2005) we show that it is actually the only way to retain the property of dynamic consistency.*

Remark 5.3. *Sarin and Wakker (1998a) show in their theorem 3.2 that dynamic consistency is equivalent to the additivity of the Choquet functional. Our theorem strengthens this result by showing that dynamic consistency on fixed filtration actually implies that the algebra generated by this filtration belongs to the algebra of N-unambiguous events and vice versa.*

6 Dynamic Characterization of Z-unambiguous events

We begin this section by presenting the 4-color experiment, suggested by Zhang (2002), and extend it to a dynamic framework. In particular it illustrates that conditionally on a Z-unambiguous event (which is not N-unambiguous) it is impossible that a consequentialist DM satisfies the property of dynamic consistency.

Example 6.1. *Consider an urn containing 100 balls. The color of each ball may be black (B), red (R), gray (G) or white (W). The DM is supposed to rank six acts, $f, f', g, g', h, h' \in \mathcal{F}$, which are defined as below. At the ex ante stage ($t = 0$) the DM is told that that the sum of black and red balls is 50 and the sum of black and gray is also 50. At interim stage ($t = 1$) one ball is drawn at random from the urn and the DM is informed the event $\{B, R\}$ occurred.*

Suppose that at the ex ante stage ($t = 0$) the DM is ambiguity averse and displays the following pattern of preferences:

$$f = \begin{pmatrix} 1 & \textit{if } \omega \in B \\ 100 & \textit{if } \omega \in R \\ 0 & \textit{if } \omega \in G \\ 0 & \textit{if } \omega \in W \end{pmatrix} \succ \begin{pmatrix} 100 & \textit{if } \omega \in B \\ 0 & \textit{if } \omega \in R \\ 0 & \textit{if } \omega \in G \\ 0 & \textit{if } \omega \in W \end{pmatrix} = f'$$

$$g = \begin{pmatrix} 1 & \textit{if } \omega \in B \\ 100 & \textit{if } \omega \in R \\ 100 & \textit{if } \omega \in G \\ 0 & \textit{if } \omega \in W \end{pmatrix} \prec \begin{pmatrix} 100 & \textit{if } \omega \in B \\ 0 & \textit{if } \omega \in R \\ 100 & \textit{if } \omega \in G \\ 0 & \textit{if } \omega \in W \end{pmatrix} = g'$$

$$h = \begin{pmatrix} 1 & \textit{if } \omega \in B \\ 100 & \textit{if } \omega \in R \\ 100 & \textit{if } \omega \in G \\ 100 & \textit{if } \omega \in W \end{pmatrix} \succ \begin{pmatrix} 100 & \textit{if } \omega \in B \\ 0 & \textit{if } \omega \in R \\ 100 & \textit{if } \omega \in G \\ 100 & \textit{if } \omega \in W \end{pmatrix} = h'$$

The DM prefers f to f' and she also prefers h and h', because the chance of getting 100 *by choosing f is the same as by choosing f', but also with additional chance of getting* 1 *under f. The same way of reasoning holds for the preference relation between the act h and the act h'. Furthermore, the DM prefers g' to g. Choosing the act g' leads to the payment of* 100 *with probability of one half, since the probability of the event $\{B, G\}$ is known to be one half, whereas the act g pays* 100 *only with probability in the range between null and one half. Moreover, changing the outcome on the event $\{G, W\}$ in the pair of acts $\{f, f'\}$ and $\{h, h'\}$ leaves the preference relation between these acts unchanged. Thus, the event $\{B, R\}$ is Z-unambiguous. In particular the collection of all Z-unambiguous events, $\mathcal{A}_Z^U = \{\emptyset, \{B, R\}, \{G, W\}, \{B, G\}, \{R, W\}, \Omega\}$, is not an algebra, since it is not closed under intersections. However, as mentioned before, it is a λ-system.*

Consider now the filtration $\mathcal{P} = \{\mathcal{P}_0, \mathcal{P}_1\}$, with $\mathcal{P}_0 = \Omega$ and $\mathcal{P}_1 = \{\{B, R\}, \{G, W\}\}$. At the interim stage ($t = 1$) the DM is informed that the event $\{B, R\}$ occurred. Since all acts $a, b \in \{f, g, h\}$ and all acts $a', b' \in \{f', g', h'\}$ are the same on the event $\{B, R\}$, $a = b$ and $a' = b'$, and differ only outside of that event, consequentialism requires that $a \sim_{\{A,B\}} b$ and $a' \sim_{\{A,B\}} b'$ and furthermore $a \succ_{\{B,R\}} a'$ (or $a \prec_{\{B,R\}} a'$ respectively). But this is possible only by reversing the conditional preference relation between g and g'. Thus, it is impossible for the ambiguity averse DM to respect dynamic consistency on the fixed filtration $\mathcal{P}$ made up of Z-unambiguous events.

We maintain dynamic consistency for all acts measurable with respect to the filtration $\mathcal{P}$. That is for all $f \in \mathcal{F}$ such that for any $x \in X$, $f^{-1}(x) \in \mathcal{P}$. Denote by $\mathcal{F}_{\mathcal{P}}$ the set of all acts measurable with respect to the filtration $\mathcal{P}$. We say that an event $A \in \mathcal{A}$ is $\mathcal{P}$ measurable if the indicator function of A is measurable with respect to the filtration $\mathcal{P}$.

Axiom 6.1 ($\mathcal{P}$-Dynamic Consistency)**.** *For any non-null event $A \in \mathcal{A}$ which is $\mathcal{P}$ measurable and for any $f, g \in \mathcal{F}_{\mathcal{P}}$, $f \sim_A g \Leftrightarrow f_A g \sim g$.*

In the same spirit as for N-unambiguous events, we look for the most natural revision rule to update capacities conditionally on Z-unambiguous events. According to the next result, applying the Bayes revision rule is the only way to ensure that the conditional certainty equivalent consistency and the $\mathcal{P}$-dynamic consistency are satisfied.

Proposition 6.1. *Let ν be a capacity on Ω and let $A \in \mathcal{A}_Z^U$ be a Z-unambiguous event, then the following two statements are equivalent:*

i) The capacity ν is updated according to the Bayes updating rule.

ii) Conditional certainty equivalent consistency and $\mathcal{P}$-dynamic consistency are satisfied.

Next we show that conditional on Z-unambiguous events the Bayes revision rule coincides with all the h-Bayesian updating rules, whenever h is a constant act, and with the Full-Bayesian updating rule.

Proposition 6.2. *Let ν be a capacity on Ω and let $A \in \mathcal{A}_Z^U$ be an Z-unambiguous event, then the Full-Bayesian updating rule and all the h-Bayesian updating rules, with $h = x$ for some $x \in X$, coincide with the Bayes updating rule.*

In the following, we assume that the finest partition in $\mathcal{P}$ contains at least three elements. Then we provide a necessary and sufficient condition for Z-unambiguous events in a conditional decision problem.

Theorem 6.1. *Let $\mathcal{P} = \{\mathcal{P}_t\}_{t \in \mathcal{T}}$ be a fixed filtration on Ω. If a decision maker has CEU preferences then the following conditions are equivalent:*

i) Conditional certainty equivalent consistency and $\mathcal{P}$-dynamic consistency are satisfied on $\mathcal{P}$.

ii) $\mathcal{A}_{\mathcal{P}}$ belongs to $\mathcal{A}_Z^U$ and ν is updated according to the Bayes updating rule.

Remark 6.1. *If conditional certainty equivalent consistency is satisfied but not $\mathcal{P}$-dynamic consistency, then the event fails to be Z-unambiguous. When updated according to the Full-Bayes updating rule, the capacities known as ϵ-contamination respect conditional certainty equivalent consistency. For a characterization of capacities which satisfies the conditional certainty equivalent consistency on all events see Eichberger, Grant, and Lefort (2009).*

Remark 6.2. *This characterization of Z-unambiguous events through conditional certainty equivalent consistency is a specific property of CEU preferences. For instance when preferences admit the multiple prior representation, then according to the result of Pires (2002) conditional certainty equivalent consistency holds on all events whenever the Full-Bayesian updating rule is used.*

7 Conclusion

In this paper the notion of unambiguous events is related to conditional decision problems. We consider a consequentialist decision maker with Choquet expected utility preferences. We look for a fixed collection of events on which the DM respects dynamic consistency. It turns out that dynamic consistency satisfied on a fixed filtration guarantees that its elements are N-unambiguous events. The converse is also true, when the capacity is updated according the Bayes updating rule. As an implication, the DM will in general violate dynamic consistency on events which are Z-unambiguous (but not N-unambiguous). However, when the fixed filtration is made up of Z-unambiguous events, the DM's preferences respect an axiom called conditional certainty equivalence consistency and dynamic consistency constrained to partition measurable acts.

On the one side, the tight structure of CEU models can be seen as a drawback of these models. On the other, side it allows to characterize sharply the usual dynamic properties of preferences from the static point of view. We hope that these results on their own may give some new insights into the nature of dynamic CEU preferences and may also contribute to the existing debate regarding the suitable notion of unambiguous events.

A Appendix

Proposition 3.1. *i)* $\Rightarrow$ *ii)* Let A be a N-unambiguous event. Suppose that there are four acts $f, f', g, g' \in \mathcal{F}$ such that $f_A g \succeq f'_A g$, but $f_A g' \prec f'_A g'$. By computing the Choquet expectations of $f_A g$ we get

$$\begin{aligned}
\int_\Omega u \circ (f_A g)\ d\nu &= u(x_1) + \sum_{j=2}^{n} [u(x_j) - u(x_{j-1})]\nu(A_j, \ldots, A_n) \\
&= u(x_1)(\nu(A) + \nu(A^c)) \\
&+ \sum_{j=2}^{n} [u(x_j) - u(x_{j-1})](\nu((A_j, \ldots, A_n) \cap A) + \nu((A_j, \ldots, A_n) \cap A^c)) \\
&= \int_A u \circ f\ d\nu + \int_{A^c} u \circ g\ d\nu.
\end{aligned}$$

Furthermore, after computing the Choquet expectations of $f'_A g$, $f_A g'$, and $f'_A g'$ we obtain

$$\int_A u \circ f\ d\nu \geq \int_A u \circ f'\ d\nu,$$

and

$$\int_A u \circ f\ d\nu < \int_A u \circ f'\ d\nu.$$

Thus, we get a contradiction.

ii) $\Rightarrow$ *i)*

Step 1. Fix an event $A \in \mathcal{A}$. For any act $f \in \mathcal{F}$ take an outcome $x \in X$ such that $f_A x \sim x$. Let $m_{\rho(f_A x)}$ be a rank dependent probability assignment for rank ρ generated by $f_A x$. Hence, $\int_\Omega u \circ (f_A x)\ d\nu = \int_\Omega u \circ (f_A x)\ dm_{\rho(f_A x)}$. Take any $y \in X$ such that $f_A x$ and $f_A y$ are comonotonic. By the Sure-Thing-Principle we have $f_A y \sim x_A y$. After computing the Choquet integral we obtain

$$\int_A u \circ f\ dm_{\rho(f_A x)} + u(x) m_{\rho(f_A x)}(A^c) = u(x),$$

thus,

$$\int_A u \circ f\ dm_{\rho(f_A x)} = u(x) m_{\rho(f_A x)}(A).$$

Furthermore, whenever $u(x) < u(y)$ we have

$$u(x) m_{\rho(f_A x)}(A) + u(y) m_{\rho(f_A x)}(A^c) = u(y)\nu(A^c) + u(x)(1 - \nu(A^c)).$$

By continuity of u there are infinitely many such outcomes y and therefore we get

$$\nu(A^c) = m_{\rho(f_A x)}(A^c).$$

Let now $B \in \mathcal{A}$ be an event such that $B = \{\omega | f(\omega) \succ x\}$, then

$$m_{\rho(f_A x)}(A^c) = \nu(A^c \cup B) - \nu(B)$$

and

$$\nu(A^c) = \nu(A^c \cup B) - \nu(B).$$

This holds for any $B \in \mathcal{A}$ such that $B \cap A^c = \emptyset$. Since the Sure-Thing-Principle is satisfied at A^c as well, then

$$\nu(A) = \nu(A \cup C) - \nu(C)$$

for any $C \in \mathcal{A}$ such that $A \cap C = \emptyset$.

Step 2. For any $x, z \in X$ such that $u(x) < u(z)$, there exists $y \in X$ with $u(x) < u(y) < u(z)$ such that $f_A g \sim f'_A g$ where the acts $f_A g$ and $f'_A g$ are defined as follows

$$f_A g = \begin{pmatrix} z & \text{if } \omega \in A \cap B \\ x & \text{if } \omega \in A \cap B^c \\ x & \text{if } \omega \in A^c \end{pmatrix} \quad \text{and} \quad f'_A g = \begin{pmatrix} y & \text{if } \omega \in A \cap B \\ y & \text{if } \omega \in A \cap B^c \\ x & \text{if } \omega \in A^c \end{pmatrix}.$$

By the Sure-Thing-Principle $f_A g \sim f'_A g \;\Rightarrow\; f_A h \sim f'_A h$ for any $f_A h$ and $f'_A h$ defined as

$$f_A h = \begin{pmatrix} z & \text{if } \omega \in A \cap B \\ x & \text{if } \omega \in A \cap B^c \\ z & \text{if } \omega \in A^c \cap B \\ x & \text{if } \omega \in A^c \cap B^c \end{pmatrix} \quad \text{and} \quad f'_A h = \begin{pmatrix} y & \text{if } \omega \in A \cap B \\ y & \text{if } \omega \in A \cap B^c \\ z & \text{if } \omega \in A^c \cap B \\ x & \text{if } \omega \in A^c \cap B^c \end{pmatrix}.$$

Now by computing the Choquet integrals, we get

$$\begin{aligned}
f_A g &= u(x)(1-\nu(A\cap B)) + u(z)\nu(A\cap B) \\
f'_A g &= u(x)(1-\nu(A)) + u(y)\nu(A) \\
f_A h &= u(x)(1-\nu(B)) + u(z)\nu(B) \\
f'_A h &= u(x)(1-\nu(A\cup(A^c\cap B))) + u(y)(\nu(A\cup(A^c\cap B)) - \nu(A^c\cap B)) \\
&\quad + u(z)\nu(A^c\cap B).
\end{aligned}$$

Since $f_A g \sim f'_A g$ we obtain

$$\begin{aligned}
u(x)(1-\nu(A\cap B)) + u(z)\nu(A\cap B) &= u(x)(1-\nu(A)) + u(y)\nu(A) \\
u(x)(\nu(A)-\nu(A\cap B)) &= u(y)\nu(A) - u(z)\nu(A\cap B).
\end{aligned}$$

From Step 1 we have $\nu(A) = \nu(A\cup(A^c\cap B)) - \nu(A^c\cap B)$ and since $f_A h \sim f'_A h$ we obtain

$$u(x)(\nu(A\cup(A^c\cap B)) - \nu(B)) = u(y)\nu(A) - u(z)(\nu(A^c\cap B) - \nu(B)).$$

Since this equation is true for any $x, z \in X$, then $\nu(B) = \nu(B\cap A) + \nu(B\cap A^c)$ for any $B \in \mathcal{A}$ and we conclude that A is a N-unambiguous event, i.e. $A \in \mathcal{A}^U_N$.

□

Proposition 5.1. *i*) ⇒ *ii*) Let us suppose that conditional certainty equivalent consistency is satisfied. Let $f = y_B x$ be a simple bet with $u(x) < u(y)$. By solvability, there is $z \in X$ such that $f \sim_A z$. Thus, by conditional certainty equivalent consistency, we have $f_A z \sim z$. After rearranging terms, we get

$$\begin{aligned}
u(z) &= u(x)(1-\nu_A(B)) + u(y)\nu_A(B) \\
u(z) &= u(x)(1-\nu(A^c\cup B)) + u(z)(\nu(A^c\cup(B\cap A)) - \nu(B)) + u(y)\nu(B\cap A).
\end{aligned}$$

Thus,

$$u(z) = \frac{u(x)(1-\nu(A^c\cup B)) + u(y)\nu(A\cap B)}{1-\nu(A^c\cup B) + \nu(A\cap B)}.$$

Since A is a N-unambiguous event, then by the property of additive separability, we get $1-\nu(A^c\cup B)+\nu(B)=1-\nu(A^c)-\nu(A\cap B)+\nu(A\cap B)=\nu(A)$. Thus, for any outcomes $x,y\in X$ such that $u(x)<u(y)$ the following is true

$$\begin{aligned} u(z) &= \frac{u(x)(1-\nu(A^c\cup B))+u(y)\nu(A\cap B)}{\nu(A)} \\ &= u(x)(1-\nu_A(B))+u(y)\nu_A(B). \end{aligned}$$

Therefore, we have

$$\nu_A(B)=\frac{\nu(A\cap B)}{\nu(A)}.$$

ii) $\Rightarrow$ *iii)* Now, suppose that the capacity ν is updated according to the Bayes updating rule. Let the events A and B be N-unambiguous. Consider acts $f,g\in\mathcal{F}$ with the following conditional preference relation: $f\prec_A g$ and $f\prec_B g$. By computing the conditional Choquet expected utilities we get

$$\begin{aligned} \int_\Omega u\circ f d\nu_A &= u(x_1)+\sum_{j=2}^{n}[u(x_j)-u(x_{j-1})]\nu_A(A_j,\dots,A_n) \\ &= u(x_{1)})+\sum_{j=2}^{n}[u(x_j)-u(x_{j-1})]\frac{\nu((A_j,\dots,A_n)\cap A)}{\nu(A)}, \\ \int_\Omega u\circ f d\nu_{A\cup B} &= u(x_1)+\sum_{j=2}^{n}[u(x_j)-u(x_{j-1})]\nu_{A\cup B}(A_j,\dots,A_n) \\ &= u(x_1)+\sum_{j=2}^{n}[u(x_j)-u(x_{j-1})]\frac{\nu((A_j,\dots,A_n)\cap(A\cup B))}{\nu(A\cup B)}. \end{aligned}$$

Since the event $A\cup B$ is N-unambiguous we have $\nu((A_j,\dots,A_n)\cap(A\cup B))=\nu((A_j,\dots,A_n)\cap A)+\nu((A_j,\dots,A_n)\cap B)$ for any $j=2,\dots,n$. Hence, the conditional Choquet integral $\int_\Omega u\circ f d\nu_{A\cup B}$ is proportional to the sum of $\int_\Omega u\circ f d\nu_A$ and $\int_\Omega u\circ f d\nu_B$. Therefore, we obtain $f\prec_{A\cup B} g$.

iii) $\Rightarrow$ *i)* Dynamic consistency directly implies conditional certainty equivalent consistency. □

Proposition 5.2. Consider the Full-Bayesian updating rule,

$$\nu_A^{FB}(B)=\frac{\nu(A\cap B)}{1-\nu(A^c\cup B)+\nu(A\cap B)}.$$

Since the conditional event A is N-unambiguous, $\nu(A^c \cup B) = \nu(A^c) + \nu(A \cap B)$ and $\nu(A^c) + \nu(A) = 1$, therefore we have

$$\nu_A^{FB}(B) = \frac{\nu(A \cap B)}{\nu(A)}.$$

Consider now the Dempster-Shafer updating rule,

$$\nu_A^{DS}(B) = \frac{\nu((A \cap B) \cup A^c) - \nu(A^c)}{\nu(A)}.$$

Since A is a N-unambiguous event, $\nu((A \cap B) \cup A^c) - \nu(A^c) = \nu(A \cap B) + \nu(A^c) - \nu(A^c)$, therefore we have

$$\nu_A^{DS}(B) = \frac{\nu(A \cap B)}{\nu(A)}.$$

Since A is N-unambiguous event, then for any $f \in \mathcal{F}$ we get

$$\begin{aligned}
\int_\Omega u \circ f d\nu &= u(x_1) + \sum_{j=2}^{n} [u(x_j) - u(x_{j-1})]\nu(A_j, \ldots, A_n) \\
&= u(x_1)(\nu(A) + \nu(A^c)) \\
&+ \sum_{j=2}^{n} [u(x_j) - u(x_{j-1})](\nu((A_j, \ldots, A_n) \cap A) + \nu((A_j, \ldots, A_n) \cap A^c)) \\
&= \int_A u \circ f \; d\nu + \int_{A^c} u \circ f \; d\nu.
\end{aligned}$$

Thus, by definition of the h-Bayesian updating rules: $f \preceq_A g$ iff $f_A h \preceq g_A h$. For a N-unambiguous event this is equivalent to $\int_A u \circ f \; d\nu \leq \int_A u \circ g \; d\nu$ which is independent of h. So all the h-Bayesian updating rules coincide when the conditional event A is N-unambiguous. □

Theorem 5.1. i) $\Rightarrow$ *ii)* Let $A \in \mathcal{A}$ be an event on which dynamic consistency is satisfied. It is well known (see Ghirardato, Maccheroni, and Marinacci (2008)) that dynamic consistency implies that the utility functions u and u_A are the same up to an affine transformation. Let $f = (A_1, x_1; \ldots; A_n, x_n)$ be an act such that $u(x_i) < u(x_{i+1})$ with $1 \leq i \leq n-1$. The Choquet expectation of f is taken with respect to a rank dependent probability assignment $m_{\rho(f)}$ with rank ρ given the act f, i.e.

$$\int_\Omega u \circ f \; d\nu = \int_\Omega u \circ f \; dm_{\rho(f)}.$$

By solvability, there is an outcome $x \in X$ such that $f \sim_A x$. Without loss of generality, we assume that f does not take the value x, i.e. $x \neq x_i$ with $i = 1, \ldots, n$. Consider acts $f_A y$ for any $y \in X$. Let $m_{\rho(f_A y)}$ be a rank dependent probability assignment associated with the act g. Let ν be a capacity such that $\nu(A) + \nu(A^c) = 1$ and let ν_A be a conditional capacity given A. In the first step we prove that

$$\frac{1}{\nu(A)} \int_A u \circ f \; dm_{\rho(f_A y)} = \int_A u \circ f \; d\nu_A.$$

In the second step, it is shown that for any act $f \in \mathcal{F}$

$$\frac{1}{\nu(A)} \int_A u \circ f \; dm_{\rho(f)} = \int_A u \circ f \; d\nu_A.$$

In the third step we conclude that that $m_{\rho(f)}(A) = \nu(A)$ for any act $f \in \mathcal{F}$. Thus, for any ranking position of states, that is for all ranks $\rho \in \mathcal{R}$, $m_\rho(A) = \nu(A)$ and therefore A is a N-unambiguous event.

Step 1. Since $f \sim_A x$, by dynamic consistency we get $f_A y \sim x_A$ for any $y \in X$.

i) Let y be an outcome such that $u(y) < u(x)$. Since $\int_\Omega u \circ g \; d\nu = \int_\Omega u \circ (f_A y) \; dm_{\rho(f_A y)}$ we have

$$\int_A u \circ f dm_{\rho(f_A y)} + u(y) m_{\rho(f_A y)}(A^c) = u(y)(1 - \nu(A)) + u(x)\nu(A).$$

This equality is true for any such outcome y for which the ranking ρ given the act $f_A y$ and the ranking ρ' given the act $x_A y$ are the same, i.e. $\rho = \rho'$. Thus, we get the following equality $u(y) m_{\rho(f_A y)}(A^c) = u(y)(1 - \nu(A))$, which implies that

$$m_{\rho(f_A y)}(A) = v(A). \qquad (1)$$

Therefore, we conclude that $\int_A u \circ f dm_{\rho(f_A y)} = u(x) m_{\rho(f_A y)}(A)$.

ii) Let y^* be an outcome such that $u(x) < u(y^*)$. Again, since $\int_\Omega u \circ (f_A y^*) \; d\nu = \int_\Omega u \circ (f_A y^*) \; dm_{\rho(f_A y^*)}$ we have

$$\int_A u \circ f \; dm_{\rho(f_A y^*)} + u(y^*) m_{\rho(f_A y^*)}(A^c) = u(y^*)(1 - \nu(A)) + u(x)\nu(A).$$

This equality is true for all outcomes y^* which keep the same ranking. Namely, the rank ρ associated with the act $f_A y^*$ and the rank ρ' associated with the act $x_A y^*$ are the same, i.e. $\rho(\omega) = \rho'(\omega)$ for all $\omega \in \Omega$. So we have the equality $u(y^*) m_{\rho(f_A y^*)}(A^c) = u(y^*)\nu(A^c)$, which implies that

$$m_{\rho(f_A y^*)}(A^c) = \nu(A^c). \qquad (2)$$

Therefore, we have $\int_A u \circ \ f dm_{\rho(f_A y^*)} = u(x)(1 - \nu(A^c))$.

iii) Consider now an act $f_A x$. Let $m_{\rho(f_A x)}$ be a rank dependent probability assignment with rank ρ given the act $f_A x$. Since the act f does not take the value x, there is an outcome $y \in X$ such that $u(y) = u(x) - \epsilon$ and there is an outcome $y^* \in X$ such that $u(y^*) = u(x) + \epsilon$ and such that the act $f_A y$ and the act $f_A y^*$ are comonotonic acts. This is possible by continuity of u. By applying (1) and (2) to $m_{\rho(f_A x)}$ we can deduce that $m_{\rho(f_A x)}(A) = \nu(A)$ and $m_{\rho(f_A x)}(A^c) = \nu(A^c)$ and therefore $\nu(A) + \nu(A^c) = 1$.

Thus, for any outcome $y \in X$ and for any rank dependent probability assignment $m_{\rho(f_A y)}$ with rank ρ given the act $f_A y$ we have

$$u(x) = \frac{1}{v(A)} \int_A u \circ f \ dm_{\rho(f_A y)} = \int_A u \circ f \ dv_A.$$

Step 2. Since $f \sim_A x$, dynamic consistency implies that $f \sim x_A f$. Let $m_{\rho(x_A f)}$ be a rank dependent probability assignment for a rank ρ given the act $x_A f$. Thus, we have

$$\int_A u \circ f \ dm_{\rho(f)} + \int_{A^c} u \circ f \ dm_{\rho(f)} = \int_{A^c} u \circ f \ dm_{\rho(x_A f)} + u(x) m_{\rho(x_A f)}(A).$$

Let us consider an act $f^* \in \mathcal{F}$ such that $f(\omega) = f^*(\omega)$ for any $\omega \in A$, but $f(\omega) \neq f^*(\omega)$ for at least one $\omega \in A^c$. Moreover, let f^* be comonotonic with f and let $x_A f$ be comonotonic with $x_A f^*$. According to dynamic consistency we have $f_A f^* \sim x_A f^*$. Therefore, we obtain the following equality

$$\int_A u \circ f \ dm_{\rho(f)} + \int_{A^c} u \circ f^* \ dm_{\rho(f)} = \int_{A^c} u \circ f^* \ dm_{\rho(x_A f)} + u(x) m_{\rho(x_A f)}(A^c),$$

which implies that $\int_A u \circ f \, dm_{\rho(f)} = u(x)m_{\rho(x_A f)}(A)$. Since dynamic consistency is satisfied on the event A, it is also satisfied on the complementary event A^c. Thus, applying step 1 to A^c we get $m_{\rho(x_A f)}(A) = \nu(A)$.

Step 3. From step 2 we have

$$u(x) = \frac{1}{v(A)} \int_A u \circ f \; dm_{\rho(f)}.$$

From step 1 we have for any $y \in X$

$$u(x) = \frac{1}{v(A)} \int_A u \circ f \; dm_{\rho(f_A y)}.$$

Therefore, we have for any $y \in X$

$$\int_A u \circ f \; dm_{\rho(f)} = \int_A u \circ f \; dm_{\rho(f_A y)}.$$

Let us consider an act g that is f measurable and comonotonic with the act f. Then, $\int_\Omega u \circ f \; dm_{\rho(f)} = \int_\Omega u \circ g \; dm_{\rho(g)}$. For any outcome y^* there is an outcome y such that $g_A y^*$ is $f_A y$ measurable and comonotonic with the act $f_A y$. By applying the same way of reasoning for act g as for act f in step 1 and in step 2 we obtain

$$\int_A u \circ g \; dm_{\rho(g)} = \int_A u \circ f \; dm_{\rho(f)} = \int_A u \circ g \; dm_{\rho(g_a y^*)} = \int_A u \circ f \; dm_{\rho(f_A y)}.$$

This implies that on the algebra on A generated by f we obtain $m_{\rho(f)} = m_{\rho(f_A y)}$. From step 1 we have that $\nu(A) = m_{\rho(f_A y)}(A)$. Therefore, we get $\nu(A) = m_{\rho(f)}(A)$ for any act $f \in \mathcal{F}$.

ii) $\Rightarrow$ *i*) See Proposition 5.1. *ii*) $\Rightarrow$ *iii*).

□

Proposition 6.1. *i*) $\Rightarrow$ *ii*) $\mathcal{P}$-Dynamic Consistency follows directly: the capacity on the filtration constructed from Z-unambiguous events is additive. Applying the Bayes updating rule on it ensures dynamic consistency for filtration measurable acts. $f \sim_A x \Leftrightarrow f_A x \sim x$ is satisfied if the updating rule is h-Bayesian with $h = x$. In proposition 6.2. we prove that all the h-Bayesian updating rules with h constant coincide on Z-unambiguous events. Since the Bayes updating rule corresponds to h-Bayesian updating rule with $h = x$, such that x is the worst possible outcome in X, the property of conditional certainty equivalent consistency holds on Z-unambiguous events, when applying this updating rule.

ii) $\Rightarrow$ *i)* Let us suppose that conditional certainty equivalent consistency is satisfied. Let $f = y_B x$ be a simple bet with $u(x) < u(y)$. By solvability there is $z \in X$ such that $f \sim_A z$. Thus, by conditional certainty equivalent consistency we have $f_A z \sim z$. After some computations we get

$$\begin{aligned} u(z) &= u(x)(1 - \nu_A(B)) + u(y)\nu_A(B) \\ u(z) &= u(x)(1 - \nu(A^c \cup B)) + u(z)(\nu(A^c \cup (B \cap A)) - \nu(B)) + u(y)\nu(B \cap A). \end{aligned}$$

Thus,

$$u(z) = \frac{u(x)(1 - \nu(A^c \cup B)) + u(y)\nu(A \cap B)}{1 - \nu(A^c \cup B) + \nu(A \cap B)}.$$

Since A is a Z-unambiguous event, then by the characterization of Z-unambiguous events, we get $1 - \nu(A^c \cup B) + \nu(B) = 1 - \nu(A^c) - \nu(A \cap B) + \nu(A \cap B) = \nu(E)$. Thus, for any outcomes $x, y \in X$ such that $u(x) < u(y)$ the following is true:

$$\begin{aligned} u(z) &= \frac{u(x)(1 - \nu(A^c \cup B)) + u(y)\nu(A \cap B)}{\nu(A)} \\ &= u(x)(1 - \nu_A(B)) + u(y)\nu_A(B). \end{aligned}$$

Therefore, we have

$$\nu_A(B) = \frac{\nu(A \cap B)}{\nu(A)}.$$

□

Proposition 6.2. From the definition of Z-unambiguous events it follows directly that all the h-Bayesian updating rules with h being constant act coincide with the Bayes updating rule. If A is observed and $B \subset A$ then the Full-Bayesian updating rule is given by

$$\nu_A^{FB}(B) = \frac{\nu(B)}{1 - \nu(B \cup A^c) + \nu(B \cap A)}.$$

Since A is Z-unambiguous then $v(A \cup E^c) = v(A) + v(E^c)$. Thus,

$$\nu_A^{FB}(B) = \frac{\nu(B)}{\nu(A)}.$$

□

Theorem 6.1. (i) ⇒ (ii). Let $\mathcal{P}$ be the fixed filtration and A_j the atoms of this filtration with $1 \leq j \leq n$. From Eichberger, Grant, and Kelsey (2007) we know that conditional certainty equivalent consistency guarantees that the same utility index u is used for conditional and unconditional preference relation. Let $f = (A_1, x_1; \ldots; A_n, x_n)$ be a $\mathcal{P}$-measurable act such that $u(x_j) < u(x_{j+1})$ with $1 \leq j \leq n-1$. The Choquet expectation of f is taken with respect to a rank dependent probability assignment $m_{\rho(f)}$ associated with the act f, i.e.

$$\int_{\Omega} u \circ f \; d\nu = \int_{\Omega} u \circ f \; dm_{\rho(f)}.$$

Let us assume that A_i^c, with $i \neq 1$ and $i \neq n$, has occurred. In the first step we show that

$$\frac{1}{m_{\rho(f)}(A_i^c)} \int_{A_i^c} u \circ f \; dm_{\rho(f)} = \int_{A_i^c} u \circ f \; dv_{A_i^c}.$$

Step 1. By solvability there is an outcome $y \in X$ such that $f \sim_{A_i^c} y$. As next we construct an act g that is comonotonic with the act f. The construction is conducted as follows. If $u(y) \leq u(x_{i-1})$, we define g on A_i^c as $g = z$ on A_n with $z \in X$ and $g = f$ otherwise. By choosing z properly, that is, such that $u(z) > u(x_n)$, we obtain g such that $g \sim_{A_i^c} x$ with $u(x_{i-1}) < u(x) < u(x_{i+1})$. By continuity of u this is possible. On the other hand, if $u(x_{i+1}) \leq u(y)$ we define another act g by decreasing x_1, such that $g \sim_{A_i^c} x$ with $u(x_{i-1}) < u(x) < u(x_{i+1})$. Then the acts f and g are comonotonic, because g is different of f only on the lowest value of f, and this lowest value of g can only be lower than the lowest value of f, or the highest value of f, and this highest value of g can only be higher than the highest value of f. Therefore, we get

$$\int_{A_i^c} u \circ g \; dv_{A_i^c} = \int_{A_i^c} u \circ g \; m_{\rho(g)},$$

where $m_{\rho(g)}$ is the rank dependent probability assignment associated with the act g. Now, we apply conditional certainty equivalent consistency and get $g_{A_i^c}x \sim x$. Since $u(x_{i-1}) < u(x) < u(x_{i+1})$, the act f and the act $g_{A_i^c}x$ are comonotonic. Thus, their Choquet integrals are computed with respect to the same measure $m_{\rho(f)}$, namely $\int_\Omega u \circ (g_{A_i^c} x)\ dv = \int_\Omega u \circ (g_{A_i^c} x)\ dm_{\rho(f)}$. Thus, we have $\int_\Omega u \circ (g_{A_i^c} x)\ dv = u(x)$. Therefore, we get

$$u(x) = \int_{A_i^c} u \circ g\ dm_{\rho(f)} + m_{\rho(f)}(A_i) u(x).$$

Finally, we obtain

$$u(x) = \frac{1}{m_{\rho(f)}(A_i^c)} \int_{A_i^c} u \circ g\ dm_{\rho(f)} = \int_{A_i^c} u \circ g\ dv_{A_i^c},$$

which is also true for the act f

$$\frac{1}{m_{\rho(f)}(A_i^c)} \int_{A_i^c} u \circ f\ dm_{\rho(f)} = \int_{A_i^c} u \circ f\ dv_{A_i^c}.$$

Step 2. We show that the above result is true for any possible permutation of the indexes $\{2, \ldots, n-1\}$ of the atoms $\{A_2, \ldots, A_{n-1}\}$. That is for any such $\mathcal{P}$-measurable act f^* the rank dependent probability assignment $m_{\rho(f^*)}$ associated with the act f^* is independent of the ranking position of the event A_i provided that $i \neq 1$ and $i \neq n$. Consider an act $f^* = (A_1, x_1^*; \ldots; A_n, x_n^*)$ such that $f^* \sim_{A_i^c} y$ for some outcome $y \in X$ and such that $u(x_i)$ is between $u(x_j^*)$ and $u(x_{j+1}^*)$. Consider also an another act $f^{**} = (A_1, x_1^{**}; \ldots; A_n, x_n^{**})$ with different rearrangements of atoms, such that $u(x_i)$ is between $u(x_j^{**})$ and $u(x_{j+1}^{**})$ and such that $f^{**} \sim_{A_i^c} y$. Let $m_{\rho(f^*)}$ and $m_{\rho(f^{**})}$ be a rank dependent probability assignment associated with the act f^*, respectively with f^{**}. By applying step 1 we obtain

$$\frac{1}{m_{\rho(f^*)}(A_i^c)} \int_{A_i^c} u \circ f\ dm_{\rho(f^*)} = \frac{1}{m_{\rho(f^{**})}(A_i^c)} \int_{A_i^c} u \circ f\ dm_{\rho(f^{**})}. \quad (1)$$

Now, we can vary the values of x_1^* and x_1^{**}, equality (1) remains true, provided that f^* and f^{**} have still the same certainty equivalent conditional on the A_i^c, i.e. there is some z such that $f^* \sim_{A_i^c} z$ and $f^{**} \sim_{A_i^c} z$. Thus, it must be true that $m_{\rho(f^*)}(A_i^c) = m_{\rho(f^{**})}(A_i^c)$. Then, we have

$$m_{\rho(f^*)}(A_i^c) = 1 - m_{\rho(f^*)}(A_i) = 1 - v(A_i \cup A_{j+1}^*, \ldots, A_n) + v(A_{j+1}^*, \ldots, A_n), \quad (2)$$

and

$$m_{\rho(f^{**})}(A_i^c) = 1 - m_{\rho(f^{**})}(A_i) = 1 - v(A_i \cup A_{j+1}^{**}, \ldots, A_n) + v(A_{j+1}^{**}, \ldots, A_n). \quad (3)$$

Equations (2) and (3) lead to the following

$$v(A_i \cup A_{j+1}^{*}, \ldots, A_n) - v(A_{j+1}^{*}, \ldots, A_n) = v(A_i \cup A_{j+1}^{**}, \ldots, A_n) - v(A_{j+1}^{**}, \ldots, A_n).$$

The last equation is true for any f. Let $A_i = E$ with $i \neq 1$ and $i \neq n$. Moreover, let $F = (A_{j+1}^{*}, \ldots, A_n)$ and let $G = (A_{j+1}^{**}, \ldots, A_n)$. The left hand side of the equation is true if $(A_i \cup A_{j+1}^{*}, \ldots, A_n) - (A_{j+1}^{*}, \ldots, A_n) \neq 1$, i.e. $v(F) \neq 0$ and $v(F \cup E) \neq 1$. The right hand side of the equation is true if $(A_i \cup A_{j+1}^{**}, \ldots, A_n) - (A_{j+1}^{**}, \ldots, A_n) \neq 1$, i.e. $v(G) \neq 0$ and $v(G \cup E) \neq 1$. Thus, we get

$$v(F \cup E) - v(F) = v(G \cup E) - v(G).$$

Step 3. Since $\mathcal{P}$-dynamic consistency holds on the algebra generated by the filtration $\mathcal{P}$ the capacity ν is additive on this algebra.

Case 1. There exists an event $F \in \mathcal{P}$ such that $v(F) \neq 0$ and $v(F \cup E) \neq 1$. Thus, by additivity of v on $\mathcal{P}$ we get $v(F \cup E) - v(F) = v(E)$. Then from the result in step 1 we conclude that $v(A \cup E) = v(A) + v(E)$ for all $A \subset E^c$.

Case 2 Suppose that there exists no such event F and then let us assume at least three atoms in $\mathcal{P}$. There exists E' and E'' in $\mathcal{P}$ such that $E = E' \cup E''$ and the complements of E' and E'' are not atoms in $\mathcal{P}$. Therefore, we can apply case 1 to them obtaining

$$\begin{aligned} \nu(F \cup E) - \nu(F) &= \nu(F \cup E' \cup E'') - \nu(F \cup E') + \nu(F \cup E') - \nu(F) \\ &= \nu(E') + \nu(E'') \\ &= \nu(E). \end{aligned}$$

Therefore, we have $v(A \cup E) = v(A) + v(E)$ for all $A \subset E^c$.

By applying step 1, step 2 and step 3 to the complementary event, E^c, we can conclude that E and E^c are Z-unambiguous events.

(ii) ⇒ *(i)*. The converse follows immediately from the Proposition 6.1.

□

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