manuscript No.
(will be inserted by the editor)

Strategic Games Beyond Expected Utility

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The date of receipt and acceptance will be inserted by the editor

Abstract This paper argues that Nash equilibrium is a solution where all strategic uncertainty has been resolved and, therefore, inappropriate to model situations that involve "ambiguity." Instead, to capture what players will do in the presence of some strategic uncertainty, takes a solution concept that is closed under best replies. It is shown that such a solution concept, fixed sets under the best reply correspondence, exists for a class of games significantly wider than those games for which generalizations of Nash equilibrium exist. In particular, this solution can do without the expected utility hypothesis.

Key words Ambiguity, fixed sets under the best reply correspondence, Nash equilibrium, non-expected utility.

1 Introduction

1.1 Expected Utility

Historically expected utility was a key vehicle to advance the analysis of strategic interaction in games and, of course, one of the main contributions of game theory to economics at large (see von Neumann and Morgenstern, 1944, Section 3). As game theory unfolded, expected utility became an essential step in many of its classical results. Among the basic propositions of game theory that use (aspects of) expected utility are the following.

Representation of simultaneous moves: Traditionally (see Kuhn, 1953; Selten, 1975) simultaneous decisions by several players are represented in the extensive form by cascading information sets, rather than by several players deciding at the same node. That not knowing what the player before her has chosen is the same for the decision maker, as deciding simultaneously with her predecessor, takes a "consequentialist" decision theory under which a compound lottery is indifferent to the associated reduced lottery. Expected utility is such a decision theory.

Extensions of the payoff function: In a non-cooperative game preferences are defined over plays. (Assume throughout that preferences are representable by a utility function.) Yet, many games incorporate chance moves. Therefore, even if all players use pure strategies, those result in a probability distribution over plays—an "outcome." To extend the payoff function from plays to probability distributions over plays (which are associated with particular pure strategy combinations), expected utility is employed. Likewise, when mixed and/or behavior strategies are introduced, expected utility provides the tool to extend the payoff function (from pure strategies) to these randomized strategies.

Harsanyi- and Selten-form: That a Bayesian game can model incomplete information, via the Harsanyi transformation (Harsanyi, 1967-8), so that either the types or the original players may be conceived as the decision makers, follows from additive separability of expected utility. Without this, how one type feels about her alternatives may depend on what other types (of the same player) do.

Kuhn's theorem: Kuhn (1953) showed that mixed and behavior strategies are equivalent if and only if the extensive form satisfies perfect recall. The proof requires the computation of conditional probability distributions and would not work if probabilities were not separable from Bernoulli utility functions.[1] For expected utility this separability holds.

Kuhn's lemma: A key step in the proof of the existence of a subgame perfect Nash equilibrium in finite extensive form games involves showing that an equilibrium of a subgame and an equilibrium of the truncation[2] can be "glued together" so as to form an equilibrium of the overall game. This was proved by Kuhn (1953, Theorem 3, p. 208) for general finite extensive form games, even though Kuhn used it only for perfect information games. His proof requires additive separability of the payoff function across plays—which expected utility satisfies.

Randomized strategies and beliefs: Under expected utility a mixed strategy for player i may be interpreted as a commonly held belief by the other players about which pure strategy player i will choose—an assessment of the strategic uncertainty. In this interpretation the Nash equilibrium property demands that, given such an assessment for each player, no player is led to revise her own assessment. Formally it does then not matter whether mixed strategies are viewed as assessments (beliefs) or whether players in fact employ random devices to select a pure strategy. Without expected utility beliefs and mixed strategies may be different objects.

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[1] In fact, already generalizing Kuhn's theorem beyond finite games raises serious technical difficulties; see Aumann, 1964.

[2] The *truncation by a subgame* is the game that results if a subgame is replaced by a terminal node at which the payoffs from (an equilibrium of) the subgame accrue to the players.

Existence of Nash equilibrium: This is probably the most important implication of expected utility. Nash's (1950) first proof of the existence of an equilibrium point for finite games relies on Kakutani's fixed point theorem. The latter requires that the best reply correspondence is upper hemicontinuous with nonempty, closed, and convex values. And this follows from expected utility, because of the linearity in probabilities. Nash's (1951) second existence proof, which employs Brouwer's fixed point theorem, uses the same property of expected utility to establish that the Nash mapping is a self-map.

All these points are of utmost importance to game theory. This paper addresses the general issue raised by the last point. It studies which solution concept for finite normal form games is appropriate and exists for theories of decision under uncertainty beyond expected utility.

Reason to go beyond expected utility is often found in experiments (most prominently Allais, 1953, and Ellsberg, 1961). In particular game experiments have motivated a growing literature on solution concepts for games played by agents with non-expected utility preferences (see e.g. Crawford, 1990; Dow and Werlang, 1994; Lo, 1996; Marinacci, 2000; Ryan, 2002; Eichberger and Kelsey, 2000, 2009, 2010; Kozhan and Zarichnyi, 2008; Glycopantis and Muir, 2008). Many of these papers modify the notion of beliefs, but otherwise stick to some version of Nash equilibrium as the solution concept.

1.2 Equilibrium and Ambiguity

As an illustrative example consider the concept of *equilibrium under ambiguity* as proposed by Eichberger and Kelsey (2009). Such an equilibrium consists of a collection of capacities, one for each player, with nonempty supports,[3] such that for each player the support of her capacity is contained in the product of the opponents' pure best replies against their capacities. (A capacity for player i is a non-additive measure on the opponents' pure strategy combinations that serves to model i's beliefs.) A pure strategy combination s such that for each player i the opponents' strategies s_{-i} belong to the support of i's capacity is called an *equilibrium strategy profile.*

Three observations about this are worthwhile. First, without further restrictions on the capacities equilibrium under ambiguity is simply a coarsening of Nash equilibrium. For, every probability distribution is a capacity. In particular, given a (mixed) Nash equilibrium σ, the probability distribution induced by the opponents' strategies σ_{-i} is a capacity for each i. By the Nash property, the support of σ_{-i} (which is uniquely defined in this case) is contained in the product of the opponents' pure best replies, for all i. Therefore, σ is an equilibrium under ambiguity. That is, without constraints on capacities, an equilibrium under ambiguity exists, whenever

[3] The appropriate definition of the support of a capacity is subject to a debate; see Dow and Werlang, 1994; Marinacci, 2000; Eichberger and Kelsey, 2009.

Nash equilibrium exists—but it may *not* involve any "ambiguity." Whether equilibrium under ambiguity exists in cases where Nash equilibrium does not, has not been studied, as far as we know.

Second, in contrast to Nash equilibrium, an equilibrium under ambiguity allows players only to use pure strategies. Capacities cannot serve as models of deliberate randomization. As a consequence, in a Matching Pennies game every pure strategy combination counts as an equilibrium strategy profile. Therefore, equilibrium under ambiguity is essentially a *set-valued* solution concept. Third, not every pure best reply of a player may be contained in the supports of the other players' capacities. That is, implicitly equilibrium under ambiguity allows for a coordination of the players' expectations, even though a capacity is meant to capture a player's "lack of confidence" (Eichberger and Kelsey, 2009, p. 17) in her beliefs. The latter aspect resembles Nash equilibrium, but is at variance with the idea that players have not resolved all strategic uncertainty. To see this, let us reconsider the basis for Nash equilibrium.

In our view Nash equilibrium is based on the following thought experiment: Assume that somehow the players have figured out "the solution" of the game. What can we—as outside observers—then say about the solution? Since the players have achieved a resolution of strategic uncertainty that we—as analysts—could not have provided, we better attribute superior intellectual capabilities to the players.[4] From that we have to conclude that they all know the solution, can trust that all the players know it, and know that they all know it, and so on. In short, we may think of the solution as commonly known among the players.[5] This implies two properties: First, we have to expect each player to play a best reply against the solution. And, second, each player will consider as possible strategy choices for her opponents (at the solution) only best replies against the solution. (Thus, when beliefs are probability distributions, only best replies against the solution can obtain positive probability.) These two conclusions lead directly to Nash equilibrium—at least when the expected utility hypothesis holds.

Admittedly, the heroic assumption underlying this thought experiment may well be violated in laboratory experiments. This may be reason to expect that in the laboratory players holds different types of beliefs, say, capacities (non-additive measures) instead of probability distributions. And indeed that is the approach taken by a prominent part of the literature. But if the initial assumption is violated, how do we arrive at a solution concept that resembles Nash equilibrium, except for the notion of beliefs? Can Nash equilibrium be "ambiguous," or can (a generalization of) Nash equilibrium

[4] In their seminal contribution von Neumann and Morgenstern (1953, p. 177) suggest to think of "players" as men of genius, like Sherlock Holmes and Professor Moriarty.

[5] Regarding preferences and rationality the weaker condition of mutual knowledge suffices, when the assessment is commonly known and players hold a common prior; see Aumann and Brandenburger, 1995. If preferences are commonly known, the common prior assumption can be dispensed with; see Polak, 1999.

model "... players that are only to some degree confident about their beliefs regarding the other players' behaviour" (Eichberger and Kelsey, 2000, p. 189)?

If players have done better than we did in figuring out the solution, it is hard to see why they should "lack confidence" or "feel ambiguity." To understand what we can say about a solution into which the players may lack confidence, we need to modify the initial assumption of the thought experiment. Yet, doing so too radically only gives another known solution concept (at least under expected utility): If players have figured out nothing at all, we should expect them to play rationalizable strategies (Bernheim, 1984; Pearce, 1984). If this is regarded as too agnostic, we need to allow for some coordination of the players' expectations, but perhaps not full coordination.

So, assume that players have "figured out something," but not a single strategy combination that can be regarded as "the solution." In particular, suppose players have found that they will choose strategies in a particular *set* of strategies, which may not be a singleton set if some "lack of confidence" is to be maintained in order to mimic the laboratory. What can we then say about this "set-valued" solution?

If for some player i there is a strategy in the candidate set that is not a best reply to any one of the opponents' strategies in the set, then player i will not use this particular strategy. It does not take much confidence on the part of the others to understand this and deduce that such a strategy for i cannot have been an element of the candidate set in the first place. Therefore, we—as analysts—have to conclude that every element of the set under scrutiny has to be a best reply against some element in the set. This is the same inclusion as with Nash equilibrium. But the lack of full coordination suggests a converse. If some strategy s is indeed a best reply against some strategy in the set, then the desire to model a "lack of confidence" demands that s is included in the solution set. In fact, given a set of strategy combinations, a player can only trust that a particular strategy will not be used, if it is not a best reply against any element in the set.

Under expected utility these considerations lead to the concept of minimal strategy subsets closed under rational behavior (CURB sets; see Basu and Weibull, 1991), and not to Nash equilibrium. Unlike the latter, CURB sets do not entirely eliminate all strategic uncertainty and are, therefore, better suited to capture what may happen in the laboratory. But they are a very different solution concept as compared to Nash equilibrium, because they are based on the reverse inclusion: A CURB set contains all best replies against itself; a Nash equilibrium is a best reply against itself, but seldom the only one. But the former inclusion appears mandatory, unless it is common knowledge among the players that a particular strategy combination will be played.

There is a sense in which set-valued solution concepts, like CURB sets, conform to the spirit of the literature on "ambiguous equilibrium." In particular, this literature treats any pure strategy in the support of equilibrium beliefs as an "equilibrium strategy," because players choose only pure strategies (exceptions are Lo, 1996, and Groes et al., 1998). Therefore, concepts like "equilibrium under uncertainty" (Eichberger and Kelsey, 2000, p. 192) are also set-valued solution concepts. And this is natural if some strategic uncertainty remains. For, if each player can only be expected to choose one particular strategy, then we are back to common knowledge of the solution and there is no room for "ambiguity." That is, any solution concept that can capture the laboratory will be a set-valued concept, like CURB sets.

The drawback of CURB sets is that they depend on the expected utility hypothesis. Beliefs are modelled as (independent) probability distributions on the opponents' strategy combinations. Moreover, without a minimality requirement CURB sets can be too large. For instance, the set of all strategy combinations is always a CURB set (though often not minimal). The latter suggests that we should insist on the exclusion of strategies that are never best replies against the solution set. Stripping away also the expected utility hypothesis then leaves a natural generalization of strict equilibrium: *fixed sets* under the best reply correspondence (Ritzberger, 1996). Those are sets of strategy combinations that satisfy two properties: (a) every element of the set is a best reply against some element in the set, and (b) every best reply against some element in the set belongs to the set. That is, fixed sets satisfy both the Nash inclusion and the CURB inclusion.

A key advantage of fixed sets is their robustness to the underlying decision theory. As long as the players' (weak) preference relations over pure strategy combinations are complete, reflexive, and transitive, fixed sets in pure strategies exist for all finite games, irrespective of the decision theory under uncertainty. This is a trivial implication of finitely many strategies. But even in mixed strategies the requirements for existence are significantly weaker than for other solution concepts.

This paper shows that the existence of fixed sets in mixed strategies only takes very weak continuity assumptions on the utility functions representing preferences over uncertain prospects. More precisely, upper semi-continuity of the direct utility function and lower semi-continuity of the value function (or indirect utility function) are sufficient to deduce the existence of a fixed set in mixed strategies. In fact, these two conditions are also necessary in the sense that if (at least) one is violated, then counterexamples can be constructed.

The plan of the paper is as follows. Section 2 states definitions and notation. Section 3 contains three examples for violations of the three hypotheses underlying the existence proof of Nash equilibrium: convex-valuedness, nonempty-valuedness, and upper hemi-continuity. Section 4 states a generalized maximum theorem which is then employed in Section 5 to characterize when (mixed) fixed sets under the best reply correspondence exist. Section 6 summarizes.

2 Preliminaries

2.1 Games

A finite n-player ($n \geq 1$) *game* $\Gamma = (S, u)$ consists of the product $S = \times_{i=1}^{n} S_i$ of the players' (finite) strategy spaces S_i and a payoff function $u = (u_1, ..., u_n) : S \to \mathbb{R}^n$ that represents the players' preferences over pure strategy combinations $s \in S$. It is assumed throughout that pure strategies do not involve any uncertainty; only mixed strategies do. The game's *mixed extension* is the infinite n-player game $\tilde{\Gamma} = (\Theta, U)$ where $\Theta = \times_{i=1}^{n} \Delta_i$ is the product of the players' mixed strategy sets $\Delta_i = \left\{\sigma_i : S_i \to \mathbb{R}_+ \left| \sum_{s_i \in S_i} \sigma_i (s_i) = 1 \right.\right\}$ and $U = (U_1, ..., U_n) : \Theta \to \mathbb{R}^n$ represents the players' preferences over the probability distributions induced by mixed strategy combinations $\sigma \in \Theta$. A mixed strategy combination $\sigma \in \Theta$ *induces* the probability distribution $p : S \to \mathbb{R}_+$ on pure strategy combinations if

$$p(s) = p(s_1, ..., s_n) = \prod_{i=1}^{n} \sigma_i (s_i) \text{ for all } s \in S.$$

The payoff function U for the mixed extension has the *expected utility* (EU) form if

$$U_i (\sigma) = \sum_{s \in S} p_\sigma (s) \, u_i (s) \tag{EU}$$

for all $i = 1, ..., n$, where $p_\sigma : S \to \mathbb{R}_+$ denotes the probability distribution on S induced by $\sigma \in \Theta$.

When players use mixed strategies, there is little point in modelling beliefs as, say, capacities rather than probability distributions. After all, if players use randomization devices they better understand the laws that govern probability. Therefore, it is implicitly assumed that players are probabilistically sophisticated in the sense of Machina and Schmeidler (1992). This assumption has no implications for the particular form of U, though.

In particular, since the point is to study deviations from EU, we do *not* assume the form (EU). Instead arbitrary functions on Θ are allowed. If pure strategy payoffs $u : S \to \mathbb{R}^n$ are given, one may want to impose that $p_\sigma (s) = 1$ implies $U(\sigma) = u(s)$ for all $\sigma \in \Theta$ and all $s \in S$, where p_σ again denotes the probability distribution induced by $\sigma \in \Theta$. But this is not a serious constraint, as $u(s)$ can be viewed as the value of U at the vertex of Θ that corresponds to $s \in S$.

Even though no restrictions are placed on U in the abstract, most examples in this paper satisfy more discipline: They are Choquet integrals with respect to a capacity derived by applying a monotone increasing transformation to the probabilities $p_\sigma (s)$ of pure strategy combinations $s \in S$. This is because Choquet integrals have turned out to be the most popular generalization of EU (see e.g. Quiggin, 1982; Schmeidler, 1989; Gilboa and Schmeidler, 1989; Sarin and Wakker, 1992), in particular when applied to strategic games (see e.g. Dow and Werlang, 1994; Lo, 1996; Ritzberger, 1996; Marinacci, 2000; Eichberger and Kelsey, 2000, 2009, 2010).

The main result will concern a far larger class of utility functions, though. It will give a characterization of a class of functions for which fixed sets under the best reply correspondence exist, in terms of weak continuity properties. For the moment denote by $\mathcal{U}$ the set of *all* functions $U : \Theta \rightarrow \mathbb{R}^n$.

2.2 Solution Concepts

For each player $i = 1, ..., n$ the *pure best reply* correspondence $\beta_i : S \twoheadrightarrow S_i$ is defined by $\beta_i (s) = \arg\max_{z_i \in S_i} u_i (s_{-i}, z_i)$ for all $s \in S$ and the *mixed best reply* correspondence $\tilde{\beta}_i : \Theta \rightarrow \Delta_i \cup \{\varnothing\}$ by $\tilde{\beta}_i (\sigma) = \arg\max_{z_i \in \Delta_i} U_i (\sigma_{-i}, z_i)$ for all $\sigma \in \Theta$, where $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n) \in S_{-i} = \times_{j \neq i} S_j$ and $\sigma_{-i} \in \Theta_{-i} = \times_{j \neq i} \Delta_j$ is analogous. The products are $\beta = \times_{i=1}^{n} \beta_i : S \rightarrow S$ and $\tilde{\beta} = \times_{i=1}^{n} \tilde{\beta}_i : \Theta \rightarrow \Theta$. Extend the best reply correspondences to sets $T \subseteq S$ resp. $\Phi \subseteq \Theta$ of strategy combinations by $\beta (T) = \cup_{s \in T} \beta (s)$ resp. $\tilde{\beta} (\Phi) = \cup_{\sigma \in \Phi} \tilde{\beta} (\sigma)$.

A *pure Nash equilibrium* is an pure strategy combination $s \in S$ such that $s \in \beta (s)$. A *mixed Nash equilibrium* is a mixed strategy combination $\sigma \in \Theta$ such that $\sigma \in \tilde{\beta} (\sigma)$. A *strict Nash equilibrium* is a strategy combination $\sigma \in \Theta$ such that $\{\sigma\} = \tilde{\beta} (\sigma)$. Under EU a strict Nash equilibrium is always a pure Nash equilibrium. Without EU this need not be true. Note that in general the defining inclusion of Nash equilibrium, $\{\sigma\} \subseteq \tilde{\beta} (\sigma)$, may be proper. This is the hallmark of Nash equilibrium, since it models the resolution of strategic uncertainty that players have achieved. If a "lack of confidence" is to be modelled, then the CURB inclusion, as added in strict Nash equilibrium, $\tilde{\beta} (\sigma) \subseteq \{\sigma\}$, needs to be satisfied, too.

A set-valued generalization of the idea of a strict Nash equilibrium is the following concept (Ritzberger, 1996): A *pure fixed set* under the best reply correspondence is a set $T \subseteq S$ such that $T = \beta (T)$. A *mixed fixed set* under the best reply correspondence is a set $\Phi \subseteq \Theta$ such that $\Phi = \tilde{\beta} (\Phi)$. A fixed set under the best reply correspondence is *minimal* if it does not properly contain another fixed set under the best reply correspondence. Clearly, if a fixed set is a singleton, then it is (minimal and) a strict Nash equilibrium. Conversely, a strict Nash equilibrium is always a minimal fixed set.

3 Three Examples

The conditions for the existence of Nash equilibrium are—in the large class of games considered here—rather restrictive. This is illustrated in the present section by giving three examples. In each of these examples one of the hypotheses of Kakutani's fixed point theorem (convex values, non-empty values, and upper hemi-continuity) is violated. They are phrased as beliefs modelled by capacities, but they could also be presented in terms of rank-dependent expected utility, as proposed by Quiggin (1982).

Example 1 Consider the 2-player game below, where the upper left entry is player 1's payoff and the lower right 2's.

	s_2^1	s_2^2
s_1^1	0 / 1	2 / 0
s_1^2	3 / 2	1 / 3

Suppose that player 1's preferences are represented by the Choquet integral with respect to the capacity $\mu(T)(\sigma) = \left(\sum_{s \in T} p_\sigma(s)\right)^2$ for all $T \subseteq S$, that is,

$$\begin{aligned} U_1(\sigma) &= 3 \cdot \mu\left(\left\{\left(s_1^2, s_2^1\right)\right\}\right) + 2 \cdot \left[\mu\left(\left\{\left(s_1^2, s_2^1\right), \left(s_1^1, s_2^2\right)\right\}\right) - \mu\left(\left\{\left(s_1^2, s_2^1\right)\right\}\right)\right] \\ &+1 \cdot \left[\mu\left(\left\{\left(s_1^2, s_2^1\right), \left(s_1^1, s_2^2\right), \left(s_1^2, s_2^2\right)\right\}\right) - \mu\left(\left\{\left(s_1^2, s_2^1\right), \left(s_1^1, s_2^2\right)\right\}\right)\right] \\ &= \mu\left(\left\{\left(s_1^2, s_2^1\right)\right\}\right) + \mu\left(\left\{\left(s_1^2, s_2^1\right), \left(s_1^1, s_2^2\right)\right\}\right) + \mu\left(\left\{\left(s_1^2, s_2^1\right), \left(s_1^1, s_2^2\right), \left(s_1^2, s_2^2\right)\right\}\right) \end{aligned}$$

Denote $\sigma_1\left(s_1^1\right) = x \in [0,1]$ and $\sigma_2\left(s_2^1\right) = y \in [0,1]$. Then player 1's payoff function may be written as

$$\begin{aligned} U_1(\sigma) &= (1-x)^2 y^2 + (x + y - 2xy)^2 + (1 - xy)^2 \\ &= 1 + 2y^2 - 6xy^2 + x^2\left(1 - 4y + 6y^2\right) \end{aligned}$$

Since $1 - 4y + 6y^2 > 0$ for all $y \in [0,1]$, the payoff function is strictly convex in x, hence, maxima exist only at the boundary. At $x = 0$ the payoff is $1+2y^2$ and at $x = 1$ it is $2 + 2y^2 - 4y$. Therefore, against $y \in [0, 1/4]$ the best reply is $x = 0$ and against $y \in [1/4, 1]$ the best reply is $x = 1$. Taking player 2 as an EU maximizer, the game resembles a Matching Pennies game, where the convex hull of player 1's two best replies at $y = 1/4$ is missing. Consequently, there exists no Nash equilibrium. Still, S constitutes a pure fixed set and indeed also a mixed fixed set under the best reply correspondence.

In Example 1 the true probabilities p_σ are distorted by applying a strictly convex (increasing) function. As a consequence player 1 dislikes randomizing and her best reply correspondence is not convex valued. The next example adds a discontinuity and illustrates the possible absence of best replies. The discontinuity comes from using a neo-additive capacity (see Cohen, 1992, and Chateauneuf et al., 2007, for axiomatizations) to distort the true probabilities. Neo-additive capacities have been explicitly argued as a means to model "... a situation where the decision-maker's beliefs are represented by the additive probability distribution π, however (s)he may lack confidence in this belief." (Eichberger and Kelsey, 2009, pp. 16) (Beliefs π in this quote are p_σ in the present paper.) In the following example the discontinuity induced by the neo-additive capacity causes a failure of upper semi-continuity for player 1's payoff function, leading to an empty-valued best reply correspondence.

Example 2 Consider the same example as before, but now suppose that player 1's preferences are represented by the Choquet integral with respect to the capacity

$$\mu\left(T\right)\left(\sigma\right)=\begin{cases}0 & \text{if } \sum_{s\in T}p_{\sigma}\left(s\right)=0\\ \delta+\left(1-\delta\right)\left(\sum_{s\in T}p_{\sigma}\left(s\right)\right)^{2} & \text{if } \sum_{s\in T}p_{\sigma}\left(s\right)>0\end{cases}$$

for some $\delta\in(0,1/2)$. Denote $\sigma_1\left(s_1^1\right)=x\in[0,1]$ and $\sigma_2\left(s_2^1\right)=y\in[0,1]$. Then player 1's payoff function can be written as

$$U_1\left(0,y\right)=\begin{cases}1 & \text{if } y=0\\ 1+2\delta+2\left(1-\delta\right)y^2 & \text{if } 0<y\end{cases}$$
$$U_1\left(1,y\right)=\begin{cases}2\delta+2\left(1-\delta\right)\left(1-y\right)^2 & \text{if } y<1\\ 0 & \text{if } y=1\end{cases},\ \text{and } x\in(0,1)\Rightarrow$$
$$U_1\left(x,y\right)=\begin{cases}1+\delta+\left(1-\delta\right)x^2 & \text{if } y=0\\ 1+2\delta+\left(1-\delta\right)\left[x^2\left(1-4y\right)+2y^2-6x\left(1-x\right)y^2\right] & \text{if } 0<y\end{cases}$$

where for $y\in(0,1)$

$$\lim_{x\searrow 0}U_1\left(x,y\right)=1+2\delta+2\left(1-\delta\right)y^2=U_1\left(0,y\right),\ \text{and}$$
$$\lim_{x\nearrow 1}U_1\left(x,y\right)=2+\delta-2\left(1-\delta\right)y\left(2-y\right)>U_1\left(1,y\right)\Leftrightarrow\delta>0,$$
$$U_1\left(0,y\right)\leq U_1\left(1,y\right)\Leftrightarrow y\leq\frac{1-2\delta}{4\left(1-\delta\right)}\in\left(0,\frac{1}{4}\right),$$
$$U_1\left(0,y\right)\geq\lim_{x\nearrow 1}U_1\left(x,y\right)\Leftrightarrow y\geq\frac{1}{4}$$

Therefore, player 1's best reply correspondence is given by

$$\tilde{\beta}_1\left(y\right)=\begin{cases}1 & \text{if } y=0\\ \varnothing & \text{if } 0<y<\frac{1}{4}\\ 0 & \text{if } \frac{1}{4}\leq y\leq 1\end{cases}$$

as an interior $x\in(0,1)$ can never be a best reply due to convexity. If player 2 is an expected utility maximizer, this game has no Nash equilibrium, because player 1's best reply is empty in the relevant region. Still, S is a pure and a mixed fixed set.

In Example 2 player 1 has no best reply in the relevant region, because the discontinuity at zero of the distortion (of the true probability p_σ) induces a payoff function that is not upper semi-continuous everywhere. The next example illustrates that a player's best reply correspondence may not be upper hemi-continuous. This is again based on a neo-additive capacity. But now the distortion has a discontinuity at 1 which causes the value function to fail *lower* semi-continuity. (This insight will prove fruitful below.)

Example 3 Consider the 2-player game below

	s_2^1	s_2^2
s_1^1	1, a	0, 1
s_1^2	a, a	a, 0

and assume that both players evaluate uncertain prospects by taking the Choquet integral with respect to the capacity

$$\mu(T)(\sigma) = \begin{cases} (1-\varepsilon)\sum_{s\in T} p_\sigma(s) & \text{if } \sum_{s\in T} p_\sigma(s) < 1 \\ 1 & \text{if } \sum_{s\in T} p_\sigma(s) = 1 \end{cases}$$

where $0 < 1-a \leq \varepsilon < 1$. Denote $\sigma_1(s_1^1) = x \in [0,1]$ and $\sigma_2(s_2^1) = y \in [0,1]$. Then the payoff functions can be written as $U_1(0,y) = U_2(x,1) = a$, for all $(x,y) \in [0,1]^2$,

$$U_1(1,y) = \begin{cases} (1-\varepsilon)y & \text{if } y<1 \\ 1 & \text{if } y=1 \end{cases}, \quad U_2(x,0) = \begin{cases} (1-\varepsilon)x & \text{if } x<1 \\ 1 & \text{if } x=1 \end{cases}$$

$$U_1(x,y)|_{0<x<1} = \begin{cases} (1-\varepsilon)(a+(y-a)x) & \text{if } y<1 \\ a+(1-a)(1-\varepsilon)x & \text{if } y=1 \end{cases}, \text{ and}$$

$$U_2(x,y)|_{0<y<1} = \begin{cases} (1-\varepsilon)(x+(a-x)y) & \text{if } x<1 \\ a+(1-a)(1-\varepsilon)(1-y) & \text{if } x=1 \end{cases}$$

Therefore, when $y=0$ resp. $y=1$ player 1's (mixed) best reply is $\tilde{\beta}_1(0) = 0$ resp. $\tilde{\beta}_1(1) = 1$, because

$$U_1(0,0) = a > U_1(x,0)|_{0<x<1} = a(1-\varepsilon)(1-x) > U_1(1,0) = 0 \text{ resp.}$$
$$U_1(1,1) = 1 > U_1(x,1)|_{0<x<1} = a+(1-a)(1-\varepsilon)x > U_1(0,1) = a$$

while for $y \in (0,1)$

$$\begin{aligned} U_1(0,y) &= a \geq 1-\varepsilon > (1-\varepsilon)\max\{a,y\} \geq U_1(x,y)|_{0<x<1} \\ &= (1-\varepsilon)(a+(y-a)x), \\ U_1(0,y) &= a \geq 1-\varepsilon > U_1(1,y) = (1-\varepsilon)y \end{aligned}$$

hence, $\tilde{\beta}_1(y) = 0$ for all $y \in (0,1)$. Similarly, at $x=0$ resp. $x=1$ player 2's best reply is $\tilde{\beta}_2(0) = 1$ resp. $\tilde{\beta}_2(1) = 0$, because

$$U_2(0,1) = a > U_2(0,y)|_{0<y<1} = a(1-\varepsilon)y > U_2(0,0) = 0 \text{ resp.}$$
$$U_2(1,0) = 1 > U_2(1,y)|_{0<y<1} = a+(1-a)(1-\varepsilon)(1-y) > U_2(1,1) = a$$

while for $x \in (0,1)$

$$\begin{aligned} U_2(x,1) &= a \geq 1-\varepsilon > (1-\varepsilon)\max\{x,a\} \geq U_2(x,y)|_{0<y<1} \\ &= (1-\varepsilon)(x+(a-x)y), \\ U_2(x,1) &= a \geq 1-\varepsilon > U_2(x,0) = (1-\varepsilon)x, \end{aligned}$$

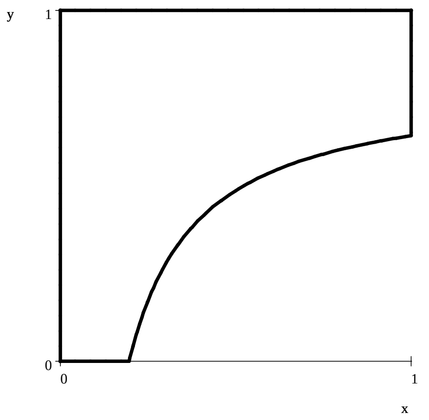

Fig. 1 The upper contour set $A(\nu)$ for $0 \leq \nu \leq (1-\varepsilon)\,a$ (Example 3).

hence, $\beta_2(x) = 1$ for all $x \in (0,1)$. That is, the best reply of player 1 (resp. 2) is constant at 0 (resp. 1), except at the point $y = 1$ (resp. $x = 1$), where it jumps to 1 (resp. 0). Best reply correspondences fail to be upper hemi-continuous at $y = 1$ resp. $x = 1$, but are otherwise continuous. Still, payoff functions are upper semi-continuous on Θ, because for $\nu \in [0,1]$ the upper contour sets (for 1's payoff function, say) are

$$\left\{(x,y) \in [0,1]^2 \,|U_1(x,y) \geq \nu\right\}$$
$$= \begin{cases} A(\nu) & \text{if } \ 0 \leq \nu \leq (1-\varepsilon)\,a \\ (\{0\} \times [0,1]) \cup ([0,1] \times \{1\}) \cup A(\nu) & \text{if } \ (1-\varepsilon)\,a < \nu \leq a \\ \left[\frac{\nu-a}{(1-\varepsilon)(1-a)}, 1\right] \times \{1\} & \text{if } a < \nu \leq 1-\varepsilon+\varepsilon a \\ \{(1,1)\} & \text{if } 1-\varepsilon+\varepsilon a < \nu \leq 1 \end{cases} \quad \text{with}$$
$$A(\nu) = \left\{(x,y) \in [0,1]^2 \,|(1-\varepsilon)\,xy + (1-\varepsilon)\,a\,(1-x) \geq \nu\right\}.$$

(In Figures 1 and 2 the upper contour set is the region enclosed by thick curves, for $0 \leq \nu \leq (1-\varepsilon)\,a$, or the region enclosed by thick curves plus the thick lines, for $(1-\varepsilon)\,a < \nu \leq a$.) That is, all upper contour sets are closed, so that U_1 is upper semi-continuous on $\Theta = [0,1]^2$. The reason why $\tilde{\beta}$ fails upper hemi-continuity is that the value function

$$V_1(y) = \max_{x \in [0,1]} U_1(x,y) = \begin{cases} a \text{ if } 0 \leq y < 1 \\ 1 \text{ if } \quad y = 1 \end{cases}$$

is only upper but not lower semi-continuous. No Nash equilibrium exists, but S is again a pure and a mixed fixed set under the best reply correspondence.

Thus, in general convex-valuedness, nonempty-valuedness, and upper hemi-continuity of the best reply correspondence may all fail. Therefore, Nash equilibrium is unlikely to exist in the large class of games at hand.

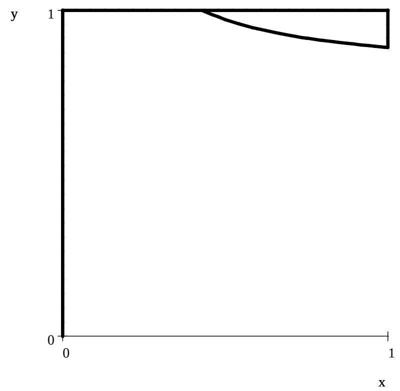

Fig. 2 The upper contour set for $(1-\varepsilon)\, a < \nu \leq a$ (Example 3).

The existence of fixed sets is much more robust, though. In order to study what precisely is needed for (mixed) fixed sets to exist we now turn to a generalized maximum theorem.

4 Maximum Theorem

Let X be a compact regular topological space and Y a topological space.[6] A real-valued function v on the product $X \times Y$ (endowed with the product topology) is *upper resp. lower semi-continuous on* X (u.s.c. resp. l.s.c. on X) if for each $w \in \mathbb{R}$ the upper contour set $\{x \in X \,|v\,(x,y) \geq w\}$ resp. the lower contour set $\{x \in X \,|v\,(x,y) \leq w\}$ is closed in X for all $y \in Y$. It is *upper resp. lower semi-continuous on* $X \times Y$ (u.s.c. resp. l.s.c. on $X \times Y$) if for each $w \in \mathbb{R}$ the set $\{(x,y) \in X \times Y \,|v\,(x,y) \geq w\}$ resp. $\{(x,y) \in X \times Y \,|v\,(x,y) \leq w\}$ is closed in $X \times Y$. Clearly, if v is u.s.c. resp. l.s.c. on $X \times Y$, then it is u.s.c. resp. l.s.c. on X, because the intersection of two closed sets is closed. A correspondence $\varphi : Y \twoheadrightarrow X$ is *upper resp. lower hemi-continuous* (u.h.c. resp. l.h.c.) if for any open set A in the range X the upper preimage $\varphi^{+}(A) = \{y \in Y \,|\varphi\,(y) \subseteq A\}$ resp. the lower preimage $\varphi^{-}(A) = \{y \in Y \,|\varphi\,(y) \cap A \neq \varnothing\}$ is open in the domain Y.

In the following statement the function v stands in for a utility function and the correspondence φ for constraints. (The constraint correspondence will be immaterial for the present paper, though, because it will be constant outside this section.)

Lemma 1 *If the function* $v : X \times Y \to \mathbb{R}$ *is u.s.c. on* X *and the correspondence* $\varphi : Y \twoheadrightarrow X$ *has a closed graph and nonempty values, then* $\sup_{x\in\varphi(y)} v\,(x,y) = \max_{x\in\varphi(y)} v\,(x,y)$ *and* $\arg\max_{x\in\varphi(y)} v\,(x,y)$ *is nonempty and compact for all* $y \in Y$*.*

[6] A topological space is *regular* if every nonempty closed set and every point that is not in the closed set can be separated by open neigborhoods.

Proof Fix $y \in Y$ and let $Q(y) = \{w \in \mathbb{R} \,|\, \exists x \in \varphi(y) : v(x,y) \geq w\}$. For each $w \in Q(y)$ the constrained upper contour set

$$H_w = \{x \in \varphi(x) \,|\, v(x,y) \geq w\} = \{x \in X \,|\, v(x,y) \geq w\} \cap \varphi(y)$$

is closed, because v is u.s.c. and $\varphi(y) \subseteq X$ is compact as a closed subset (as φ has a closed graph) of a compact space. Hence, the collection $\{H_w \,|\, w \in Q(y)\}$ has the finite intersection property, because for any finite set of numbers $w_1, ..., w_k$ with $w_h \leq w_{h+1}$ for $h = 1, ..., k-1$, say, the set H_{w_k} is contained in all the others. Since $\varphi(y) \subseteq X$ is compact, the intersection $\cap_{w \in Q(y)} H_w$ is compact and nonempty. But this intersection contains only maximizers. *Q.E.D.*

That u.s.c. on X is also necessary for the existence of a maximizer is easily seen from the example $v(x,y) = x$ for all $x \in [0,1)$ and $v(1,y) = 0$ for all $y \in Y$ (with $X = [0,1]$ and Y arbitrary), where no maximizer exists.

Whenever v is u.s.c. on X and φ has a closed graph with nonempty values, Lemma 1 enables the definition of a *value function* (or indirect utility function) $V : Y \to \mathbb{R}$ defined by

$$V(y) = \max_{x \in \varphi(y)} v(x,y) \text{ for all } y \in Y. \tag{1}$$

Leininger (1984) and Ausubel and Deneckere (1993) present generalizations of Berge's (1963, p. 116) maximum theorem.[7] Their common hypothesis is an u.s.c. (on $X \times Y$) objective function. Leininger assumes a continuous constraint correspondence with nonempty compact values and adds "graph continuity" (of the objective function) with respect to the constraint correspondence. Ausubel and Deneckere assume an u.h.c. constraint correspondence with nonempty compact values and add l.h.c. of the correspondence giving everything below attainable values (see Proposition 1(b) below). It will now be shown that the "lower part" of Leininger's graph continuity and the condition by Ausubel and Deneckere are equivalent—and, in fact, equivalent to the simpler condition that the value function is l.s.c.

The function $v : X \times Y \to \mathbb{R}$ is *lower graph continuous at* $(x,y) \in \text{graph}(\varphi) \subseteq X \times Y$ (with respect to the constraint correspondence $\varphi : Y \twoheadrightarrow X$) if for all $\varepsilon > 0$ there is a neighborhood O_ε of y in Y such that

$$\text{if } y' \in O_\varepsilon \text{ then there is } x' \in \varphi(y') \text{ such that } v(x',y') > v(x,y) - \varepsilon \tag{2}$$

It is *lower graph continuous on* graph(φ) with respect to φ (henceforth l.g.c.) if it is lower graph continuous (w.r.t. φ) at all $(x,y) \in \text{graph}(\varphi)$.[8] The next result says that lower graph continuity, the condition by Ausubel and Deneckere, and l.s.c. of the value function V are all equivalent.

[7] Walker (1979) provides a generalization of the maximum theorem by replacing maximization with a dominance relation.

[8] Leininger's original definition applies to metric spaces and replaces the inequality in (2) by $|v(x',y') - v(x,y)| < \varepsilon$. The function $v(x,y) = y$ for all $y \in [0,1)$ and $v(x,1) = 0$ for $X = Y = [0,1]$ and $\varphi(y) = X$ for all $y \in Y$ is l.g.c., but fails Leininger's definition.

Proposition 1 *If $v : X \times Y \to \mathbb{R}$ is u.s.c. on X and $\varphi : Y \twoheadrightarrow X$ has a closed graph and nonempty values, then the following three statements are equivalent:*

(a) v is l.g.c. (w.r.t. φ) on $\operatorname{graph}(\varphi)$*;*

(b) the correspondence $G : Y \twoheadrightarrow \mathbb{R}$, defined by $G(y) = \{w \in \mathbb{R} \,|\, w \le V(y)\}$ for all $y \in Y$, is l.h.c.;[9]

(c) the value function $V : Y \to \mathbb{R}$, defined by (1), is l.s.c.

Proof "(a) implies (c):" If v is l.g.c., then it is l.g.c. at $(x, y) \in \operatorname{graph}(\varphi)$ with $v(x, y) = V(y)$. By l.g.c., for any $\varepsilon > 0$ there is a neighborhood O_ε of $y \in Y$ such that $y' \in O_\varepsilon$ implies $\exists x' \in \varphi(y) : v(x', y') > v(x, y) - \varepsilon = V(y) - \varepsilon$. Since $V(y') \ge v(x', y')$, it follows that $V(y') > V(y) - \varepsilon$. This implies that V is l.s.c.

For, suppose to the contrary that there are $w \in \mathbb{R}$ and a net $\{y_t\}_{t \in D}$, for a directed set $(D, \ge)$, such that $V(y_t) \le w$ for all $t \in D$ and y_t converges to $y_0 \in Y$, but $V(y_0) > w$. Let $\varepsilon = (V(y_0) - w)/2 > 0$. By hypothesis $V(y_t) > V(y_0) - \varepsilon = (V(y_0) + w)/2 > w$ for all $t \ge d$ for some $d \in D$, in contradiction to $V(y_t) \le w$ for all t. Therefore, for any $w \in \mathbb{R}$ the lower contour set $\{y \in Y \,|\, V(y) \le w\}$ is closed, i.e., the function V is l.s.c.

"(c) implies (b):" First, G is down-closed, i.e. $w \in G(y)$ and $w' < w$ imply $w' \in G(y)$ for all $y \in Y$. This implies that, for any open set $A \subseteq \mathbb{R}$, the lower preimage $G^-(A) = \{y \in Y \,|\, G(y) \cap A \ne \varnothing\}$ is given by

$$G^-(A) = \{y \in Y \,|\, \inf(A) < V(y)\}.$$

But the latter is precisely the complement of the lower contour set for V, $\{y \in Y \,|\, V(y) \le \inf(A)\}$, which is closed if V is l.s.c. Therefore, that V is l.s.c. implies that G is l.h.c.

"(b) implies (a):" G is l.h.c. if and only if for every open set $A \subseteq \mathbb{R}$ the lower preimage $G^-(A) = \{y \in Y \,|\, G(y) \cap A \ne \varnothing\} = \{y \in Y \,|\, \inf(A) < V(y)\}$ is open. Let $(x, y) \in \operatorname{graph}(\varphi)$, $\varepsilon > 0$, and $w > V(y)$. Then the set $O_\varepsilon = G^-((v(x, y) - \varepsilon, w))$ is a neighborhood of y. If $y' \in O_\varepsilon$, then with $x' \in \varphi(y)$ such that $v(x', y') = V(y')$ it follows that $v(x', y') > v(x, y) - \varepsilon$, i.e. v is l.g.c. (w.r.t. φ). *Q.E.D.*

For the following generalized maximum theorem it is assumed that v is u.s.c. on the whole product $X \times Y$, and not only on X. The statement follows more or less directly from Leininger's (1984) result or from Theorem 2 of Ausubel and Deneckere (1993, p. 102) in combination with Proposition 1. In the Appendix we give a proof of Theorem 1 because of a few technical details. Unlike Leininger we do not assume metric spaces. Also unlike Leininger and Ausubel and Deneckere, we do not assume that the constraint correspondence φ is u.h.c., but assume directly a closed graph. That φ has a closed graph is equivalent to assuming φ u.h.c. with closed values if X is

[9] Ausubel and Deneckere (1993) define G by $G(y) = \{w \in \mathbb{R} \,|\, \exists x \in X : w \le f(x, y)\}$. But if f is u.s.c. on X this is the same as the present definition.

Hausdorff (T_2) (see Aliprantis and Border, 2006, p. 561), which will indeed be the case in the application below. Still, even if X is not Hausdorff, that φ is u.h.c. with nonempty and closed values would be an alternative hypothesis for Theorem 1. For, this implies that φ has a closed graph, because the values are compact as closed subsets of a compact space and X is regular by assumption.

Theorem 1 *If $v : X \times Y \to \mathbb{R}$ is u.s.c. on $X \times Y$, the constraint correspondence $\varphi : Y \twoheadrightarrow X$ has a closed graph and nonempty values, and the value function $V : Y \to \mathbb{R}$ defined by (1) is l.s.c. on Y, then the correspondence $\gamma : Y \twoheadrightarrow X$, defined by $\gamma(y) = \{x \in \varphi(y) \mid v(x,y) \geq v(x',y),\ \forall x' \in \varphi(y)\}$ for all $y \in Y$, is u.h.c. with nonempty compact values, and the value function V is continuous.*

The l.s.c. property for the value function V is also necessary for the maximum theorem. This follows from Example 3. In that example the best reply correspondence is not u.h.c., because the value function is not l.s.c. (while the direct utility function is u.s.c.). That the u.s.c. condition on the product $X \times Y$ is necessary for the maximum theorem, too, is shown by the following example.

Example 4 Let $X = Y = [0,1]$ and $v(x,y) = x$ for all $y \in (0,1]$, but $v(x,0) = (1-x)/2$. This function is u.s.c. (in fact continuous) on X, but not on $X \times Y$. The maximizers are $\gamma(y) = 1$ for all $y \in (0,1]$ and $\gamma(0) = 0$. Hence, γ is not u.h.c., even though the value function, $V(y) = 1$ for all $y \in (0,1]$ and $V(0) = 1/2$, is l.s.c. on Y.

5 Fixed Sets

The examples in Section 3 have shown that existence of Nash equilibrium is rare in the large class of games at hand. This is in contrast to fixed sets under the best reply correspondence. Irrespective of the players' attitudes towards uncertainty or ambiguity, a pure fixed set always exists, and so does a minimal pure fixed set. This is a trivial consequence of finiteness.

Whether a (minimal) *mixed* fixed set exists is a different matter, though. This is because a pure fixed set need not be a mixed fixed set, not even under EU. For instance, in a standard Matching Pennies game with EU preferences the unique pure fixed set is S, while the unique mixed fixed set is Θ. In this example the pure fixed set still "spans" the mixed fixed set. In the following example—without EU—the situation is worse.

Example 5 Consider the 2-player Matching Pennies game below, where again the upper left entry is $u_1(s)$ and the lower right is $u_2(s)$.

	s_2^1	s_2^2
s_1^1	0, 1	2, 0
s_1^2	3, 2	1, 3

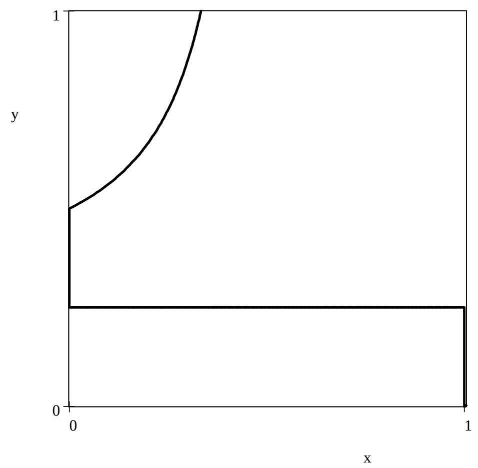

Fig. 3 Player 1's best reply correspondence from Example 5.

Assume that player 1's preferences are represented by the sum of expected utility plus its variance, i.e.

$$\begin{aligned} U_1(x,y) = \ & 1 + x + 2y - 4xy + [x + 4y - 8xy - x^2 - 4y^2 + 16xy^2 + 8x^2y \\ - \ & 16x^2y^2] = 1 + 6y - 4y^2 + 2(1-4y)(1-2y)x - (1-4y)^2 x^2 \end{aligned}$$

where $\sigma_1(s_1^1) = x \in [0,1]$ and $\sigma_2(s_2^1) = y \in [0,1]$, and the term in square brackets is the variance of 1's payoff. Since $\partial^2 V_1/\partial x^2 = -2(1-4y)^2 \leq 0$ the payoff function V_1 is strictly concave in x except at $y = 1/4$, where it is constant. Player 1's mixed best replies are

$$\tilde{\beta}_1(x,y) = \begin{cases} 1 & \text{if } 0 \leq y < \frac{1}{4} \\ [0,1] & \text{if } \quad y = \frac{1}{4} \\ 0 & \text{if } \frac{1}{4} < y < \frac{1}{2} \\ \frac{1}{2} + \frac{1}{2(1-4y)} & \text{if } \frac{1}{2} \leq y \leq 1 \end{cases}$$

(see Figure 3). Let player 2 be an expected utility maximizer. Then, even though S is a pure fixed set, it is not a mixed fixed set, because $\tilde{\beta}(0,1) = (1/3, 0)$. Still there exists a mixed fixed set, namely the set Φ that consists of the four elements $(1/3, 0)$, $(1, 0)$, $(1, 1)$, and $(1/3, 1)$. There is also a Nash equilibrium, $(x, y) = (1/2, 1/4)$. But that is neither a mixed fixed set nor a strict Nash equilibrium, because $\tilde{\beta}(1/2, 1/3) = [0,1]^2$.

In Example 5 there still exists a mixed fixed set. But fixed sets do not always exist. In the last example the best reply correspondence is not u.h.c. and no mixed fixed set exists.

Example 6 Consider the 2-player game

	s_2^1	s_2^2
s_1^1	1, 1	-1, $5/4$
s_1^2	0, $1/2$	0, 0

where for $x = \sigma_1\left(s_1^1\right) \in [0,1]$ and $y = \sigma_2\left(s_2^1\right) \in [0,1]$ the payoffs in the mixed extension are

$$\begin{aligned} U_1(x,y) &= 2xy - x^2 \text{ and} \\ U_2(x,y) &= \frac{1}{4}xy + \frac{5}{4}x - \frac{1}{2}y^2 + \theta(y - xy) \end{aligned}$$

with the function $\theta : [0,1] \to [0,1]$ given by $\theta(p) = 0$ for all $p \in [0,1)$ and $\theta(1) = 1$. Player 1's payoff function is continuous in $x \in [0,1]$ and attains its maximum at $x = \tilde{\beta}_1(y) = y$. Player 2's payoff function is continuous everywhere except at $(x,y) = (0,1)$, where it takes the value $1/2$. Therefore,

$$\tilde{\beta}_2(x) = \begin{cases} 1 & \text{if } x = 0 \\ x/4 & \text{if } x > 0 \end{cases}$$

that is, 2's best replies are not u.h.c., because her value function,

$$V_2(x,y) = \begin{cases} 1/2 & \text{if } x = 0 \\ x^2/32 + 5x/4 & \text{if } x > 0 \end{cases}$$

is not l.s.c. Still, 2's payoff is u.s.c. on $\Theta = [0,1]^2$, because for any $\nu \in [0,5/4]$ the upper contour set $\left\{(x,y) \in [0,1]^2 \,|U_2(x,y) \geq \nu\right\}$ is the closed set

$$A(\nu) = \left\{(x,y) \in [0,1]^2 \,\left|2y^2 - xy - 5x + 4\nu \leq 0\right.\right\}$$

if $\nu > 1/2$ and the disjoint union $A(\nu) \cup \{(0,1)\}$ if $\nu \leq 1/2$, where $(0,1) \notin A(\nu)$.

Suppose there is a nonempty fixed set $\Phi \subseteq \Theta$. Then the projection of Φ on player 2's coordinate y must be contained in $(0,1/4] \cup \{1\}$, because neither $y = 0$ nor $y \in (1/4,1)$ can ever be best replies for player 2. Therefore, the projection of Φ on player 1's coordinate x must also be contained in $(0,1/4] \cup \{1\}$, because player 1 always imitates player 2. Hence $\Phi \subseteq (0,1/4]^2 \cup \{(1,1)\}$ and, in particular, $(0,0) \notin \Phi$. But if $(0,0) \notin \Phi$, then $y = 1$ cannot belong to the projection of Φ on 2's coordinate and, therefore, $(1,1) \notin \Phi$ (as player 1 only chooses $x = 1$ if $y = 1$), so that $\Phi \subseteq (0,1/4]^2$. Since $\tilde{\beta}(x,y) = \{(y,x/4)\}$ for any $(x,y) \in (0,1/4]^2$, that $(x,y) \in \Phi$ implies $(4y,x) \in \Phi$. But $(4y,x) \in \Phi$ implies $(4x,4y) \in \Phi$ which implies $(16y,4x) \in \Phi$ which implies $(16x,16y) \in \Phi$, and so on. Therefore, $(x,y) \in \Phi$ implies $(4^t x, 4^t y) \in \Phi$ for all $t = 1,2,...$, yet $(4^t x, 4^t y) \leq (1/4,1/4)$ for all t implies $x = y = 0$ in contradiction to $(0,0) \notin \Phi$. Consequently there cannot be any nonempty fixed set.[10]

It is the failure of u.h.c. caused by a value function that is not l.s.c. that eliminates any fixed set in the previous example. This suggests that the u.h.c. property is indeed required for the existence of mixed fixed sets.

Hence, to study this issue the generalized maximum theorem from the previous section gets employed. All the topological assumptions are fulfilled for mixed strategies in finite games. For each player i her set Δ_i of mixed strategies is a compact subset of Euclidean space, and so is the set $\Theta_{-i} = \times_{j \neq i} \Delta_j$ of the opponents' strategies. The (constant) constraint correspondence, $\varphi_i(\sigma_{-i}) = \Delta_i$, trivially has a closed graph. Therefore, if U_i is u.s.c. on Δ_i, the value function $V_i : \Theta_{-i} \to \mathbb{R}$ can be defined by

$$V_i(\sigma_{-i}) = \max_{\sigma_i \in \Delta_i} U_i(\sigma_{-i}, \sigma_i) \tag{3}$$

for all players $i = 1, ..., n$ due to Lemma 1. For the following statement recall that $\mathcal{U}$ denotes the set of all payoff functions $U : \Theta \to \mathbb{R}^n$, not only the u.s.c. functions.

Theorem 2 *There exists a nonempty compact mixed fixed set under the best reply correspondence for every finite game and every utility function U in the class $\mathcal{U}_0 \subset \mathcal{U}$ if and only if $\mathcal{U}_0$ is the set of payoff functions such that U is u.s.c. on Θ and each V_i is l.s.c. on Θ_{-i}.*

Proof "if:" If $U \in \mathcal{U}_0$, that is, U is u.s.c. and each V_i is l.s.c., then the associated mixed best reply correspondence $\tilde{\beta} : \Theta \twoheadrightarrow \Theta$ is u.h.c. with nonempty compact values by Theorem 1. Extend $\tilde{\beta}$ to subsets $\Phi \subseteq \Theta$ by $\tilde{\beta}(\Phi) = \cup_{\sigma \in \Phi} \tilde{\beta}(\sigma)$. The claim is that there exists a subset $\Phi \subseteq \Theta$ such that $\Phi = \tilde{\beta}(\Phi)$. But this follows directly from Theorem 8 of Berge (1963, p. 113). This theorem shows that a fixed set can be found by iterating $\tilde{\beta}$, that is, $\tilde{\beta}^0(\Theta) = \Theta$ and $\tilde{\beta}^t(\Theta) = \tilde{\beta}\left(\tilde{\beta}^{t-1}(\Theta)\right)$ for all $t = 1, 2, ...$ yields $\Phi = \tilde{\beta}(\Phi) = \cap_{t=1}^{\infty} \tilde{\beta}^t(\Theta) \neq \varnothing$. That Φ is compact follows because the image of a compact set under an u.h.c. correspondence with nonempty compact values is compact.

"only if:" If U_i is not u.s.c. on Δ_i, the single player game with $\Delta_1 = [0,1]$ and a payoff function defined by $U(\sigma) = \sigma$ for all $\sigma \in [0,1)$ and $U(1) = 0$ has no fixed set, because there is no best reply. If U_i is u.s.c. on Δ_i, but not on Θ, Example 4 shows that $\tilde{\beta}_i$ may not be u.h.c. If $\tilde{\beta}_i$ is not u.h.c. or V_i is not l.s.c., then Example 6 shows that a fixed set need not exist. Therefore, that U_i is u.s.c. on Θ and V_i is l.s.c. on Θ_{-i} is also necessary in the sense that without these hypotheses counterexamples can be constructed. *Q.E.D.*

Since fixed sets may be large, there is interest in minimal fixed sets (that do not properly contain other fixed sets). Those may give sharper predictions for the laboratory.

[10] There is no Nash equilibrium either, but that is not the point of the example.

Corollary 1 *For every $U \in U_0$ and every game there exists a minimal mixed fixed set under the best reply correspondence.*

Proof The collection of all fixed sets is nonempty by Theorem 2 and partially ordered by set inclusion. Let $\{\Phi_k\}_k$, for k in a directed index set $(K, \geq)$, be a chain of mixed fixed sets such that $h \geq k$ implies $\Phi_h \subseteq \Phi_k$ for all $h, k \in K$. As fixed sets are compact by Theorem 2, the finite intersection property implies that $\Phi = \cap_{k \in K} \Phi_k$ is nonempty and compact. Since the chain is completely ordered by set inclusion, $\Phi \subseteq \Phi_k$ for all $k \in K$. Therefore, $\tilde{\beta}(\Phi) \subseteq \tilde{\beta}(\Phi_k) = \Phi_k$ for all $k \in K$, hence, $\tilde{\beta}(\Phi) \subseteq \Phi$. It remains to show that $\Phi \subseteq \tilde{\beta}(\Phi)$. Choose $\sigma \in \Phi$. Then for each $k \in K$ there is some $\sigma^k \in \Phi_k$ such that $\sigma \in \tilde{\beta}\left(\sigma^k\right)$. Since Θ is compact, for the net $\left\{\sigma^k\right\}$ there is a subnet $\left\{\sigma^h\right\}$ that converges to some $\sigma^0 \in \Theta$. Because $\sigma^h \in \Phi_k$ for all $h \geq k$, the subnet $\left\{\sigma^h\right\}$ is contained in Φ_k. Since Φ_k is compact, it follows that $\sigma^0 \in \Phi_k$ for all k. But then $\sigma^0 \in \cap_{k \in K} \Phi_k = \Phi$. Furthermore, $\sigma \in \tilde{\beta}\left(\sigma^k\right)$ for all k, $\left\{\sigma^h\right\}$ converges to σ^0, and the constant net $\{\sigma\}$ converges trivially to σ, that is, $\left(\sigma, \sigma^h\right) \rightarrow \left(\sigma, \sigma^0\right)$ and $\left(\sigma, \sigma^h\right)$ belongs to graph $\left(\tilde{\beta}\right) = \left\{(\sigma, \sigma') \in \Theta \times \Theta \,\middle|\, \sigma \in \tilde{\beta}(\sigma')\right\}$ for all h. Since $\tilde{\beta}$ is u.h.c. with nonempty closed values and Θ is regular, graph $\left(\tilde{\beta}\right)$ is closed, hence, $\sigma \in \tilde{\beta}\left(\sigma^0\right)$. Therefore Φ is itself a fixed set and a lower bound for the chain $\{\Phi_k\}$. Since this holds for any chain of fixed sets, Zorn's lemma implies that the collection of all fixed sets has a minimal element. *Q.E.D.*

Most axiom systems characterizing decision theories under uncertainty will contain some continuity axiom. After all, this is one of the main sufficient conditions for a representation of preferences by a utility function. If the continuity axiom is good enough to make U_i u.s.c. on Θ and V_i l.s.c. on Θ_{-i}, mixed fixed sets (and their minimal versions) will exist.

6 Conclusions

Nash equilibrium is built on the idea that players have resolved all strategic uncertainty. This is often unrealistic, in particular in the laboratory. To accommodate this, solution concepts for games have been proposed that extend the spirit of Nash equilibrium. By modelling players' beliefs as measures that are not probability distributions, they aim to capture how players will behave when they entertain doubts about the solution.

This paper argues that for such a situation versions of Nash equilibrium are inappropriate. For, the first thing that players should lose confidence in is that some opponent will not play a particular best reply against the solution (one that the equilibrium beliefs exclude). Consequently, a solution under a "lack of confidence" must include all best replies against the solution. But this is the reverse inclusion as under Nash equilibrium.

Another issue is that the expected utility hypothesis may be violated in the laboratory. This also affects Nash equilibrium, because the traditional existence proofs do invoke expected utility. Without the linearity in probabilities and the multiplicative separability between probabilities and Bernoulli utility the hypotheses of the commonly used fixed point theorems may fail. So, Nash equilibrium is both inappropriate and may not exist when players lack confidence.

Therefore, we propose an alternative solution concept that is a set-valued generalization of strict equilibrium: fixed sets under the best reply correspondence. This concept is closed under best replies and thereby repairs the deficiency of the other solution theories. Moreover, in pure strategies it always exists, independently of the decision theory under uncertainty. But also in mixed strategies the requirements for the existence of fixed sets are very mild indeed. Weak continuity assumption suffice to establish existence of fixed sets and their minimal versions.

7 Appendix

Proof of Theorem 1 Because v is u.s.c. on X and $\varphi(y) \subseteq X$ is compact as a closed subset of a compact space, $\gamma(y)$ is nonempty and compact for all $y \in Y$ by Lemma 1. That V is continuous follows from Theorem 2 of Berge (1963, p. 116) which states that if v is u.s.c. on $X \times Y$, then V is u.s.c. Since a function that is both u.s.c. and l.s.c. is continuous, V is continuous.

To establish that γ is u.h.c. we need to show that the upper preimage $\gamma^+(A) = \{y \in Y \,|\gamma(y) \subseteq A\}$ is open in Y for any open set $A \subseteq X$. Let $A \subseteq X$ be open and consider the complement $Y \setminus \gamma^+(A) = \{y \in Y \,|\gamma(y) \setminus A \neq \varnothing\}$. For a directed set $(D, \geq)$ let $\{y_t\}_{t \in D}$ be a net that converges to $y \in Y$ such that $y_t \in Y \setminus \gamma^+(A)$ for all $t \in D$. Then for each $t \in D$ there is $x_t \in \gamma(y_t) \setminus A$, so that $v(y_t, x_t) = V(y_t)$. Because X is compact, there is a subnet $\{(x_d, y_d)\}_{d \in D'}$ such that (x_d, y_d) converges to (x, y) for some $x \in X$ and $x_d \in \gamma(y_d) \setminus A$ for all $d \in D'$. Since φ has a closed graph by hypothesis, $x \in \varphi(y)$. Because A is open, the complement $X \setminus A$ is closed and, therefore, $x \notin A$. That V is continuous implies $\lim_{t \in D'} V(y_t) = V(y)$. Therefore, it follows from $\lim_{d \in D'} v(x_d, y_d) = V(y) \leq v(x, y)$, by the u.s.c. property of v on $X \times Y$, that $x \in \gamma(y)$. This says that $Y \setminus \gamma^+(A)$ is closed or, equivalently, that $\gamma^+(A)$ is open, so that γ is u.h.c. *Q.E.D.*

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