Overview

1. Weaknesses of NE
   - Example 1: Centipede Game
   - Example 2: Matching Pennies
2. Logit QRE
   - Explaining the Centipede Game using logit-QRE
   - Explaining Matching Pennies using Logit QRE

Literature

- Fey, M., McKelvey, R. D., and Palfrey, T. R. (1996): An Experimental Study of Constant-Sum Centipede Games, International Journal of Game Theory, 25, 269-287.
- Goeree, J. K. and Holt, C. A. (2001): Ten Little Treasures of Game Theory and Ten Intuitive Contradictions, American Economic Review, 91(5), 1402-1422
- McKelvey, R. D., and Palfrey, T. R. (1992): An Experimental Study of the Centipede Game, Econometrica, 60(4), 803-836

Nash Equilibrium

- Nash Equilibrium (NE) assumes
  - agents have correct beliefs about other agent's behavior
  - agents best respond given their beliefs
- These assumptions are often violated
  - unobservable variables (weather, temperature, framing...)
  - cognitive limitations, lack of attention/willpower
  - social preferences
  - etc.

“Improve on NE?”

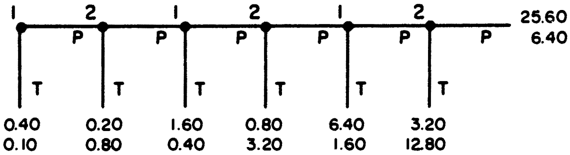
- Critics of NE typically point out that it is not always congruent with reality
- But is this the only criterion that a “good” theory has to satisfy?

"Improve on NE?"

- Critics of NE typically point out that it is not always congruent with reality
- But is this the only criterion that a "good" theory has to satisfy?
- Stigler (1965) proposes 3 criteria to evaluate economic theories:
  - Congruence with Reality
  - Tractability
  - Generality

Centipede Game

Centipede Game: McKelvey and Palfrey (1992)

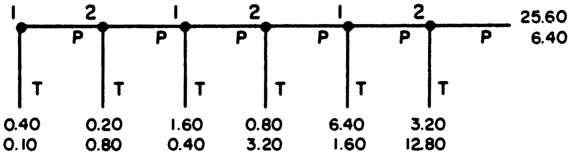


- What is the unique subgame-perfect equilibrium of this game?
  - Play T at all nodes

What are the NE?

Centipede Game

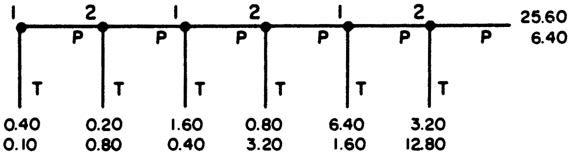
Centipede Game: McKelvey and Palfrey (1992)



- What is the unique subgame-perfect equilibrium of this game?
  - Play T at all nodes
- How many pairs of subjects play T at the first node?

Centipede Game

Centipede Game: McKelvey and Palfrey (1992)



- What is the unique subgame-perfect equilibrium of this game?
  - Play T at all nodes
- How many pairs of subjects play T at the first node?
  - 0.7%

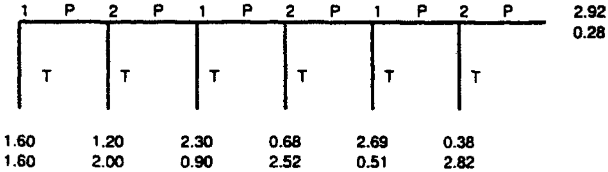
Centipede Game

- Why do subjects fail to play the subgame-perfect NE?

Centipede Game

- Why do subjects fail to play the subgame-perfect NE?
- Concern for Efficiency?
- Social Preferences?
- How could we modify the design if we want to rule out these motives?

Centipede Game



- This experiment is reported in Fey et al. (1996)
- Social preferences or efficiency concerns are unlikely to play an important role in this version

Centipede Game: Experimental Design

- 3 sessions with 18-20 subjects each
- 10 periods per session
- 3 different universities: Caltech, University of Iowa, PCC

Centipede Game: Results

7	.62	.31	.07	0	0	0	0
CIT-6	(62)	(31)	(7)	0	0	0	0
8	.77	.23	0	0	0	0	0
UI-6	(77)	(23)	0	0	0	0	0
9	.33	.48	.15	.02	.01	0	0
PCC-6	(27)	(39)	(12)	(2)	(1)	0	0
Pooled	.59	.33	.07	.007	.003	0	0
6 move	(166)	(93)	(19)	(2)	(1)	0	0

- The table contains the fraction of 281 subject pairs playing T for each possible node. In parentheses: Number of observations for each node.

Matching Pennies

- Just like in the centipede game, matching pennies has only one Nash equilibrium. As a result, it is suitable to test whether NE is consistent with reality.
- Moreover, the NE in matching pennies is mixed
  - It is not obvious that people are able to randomize
    - When asked to produce random sequences, subjects typically produce sequences with too few long runs.
    - It is not easy to find out how another player randomizes - unless there is repeated interaction. Therefore, it is sufficient if players *believe* the other player randomizes using equilibrium probabilities
  - Instead of assuming individual players are randomizing, we could assume that different types of players play different pure strategies and that the population consists of fractions of types that correspond to the equilibrium probabilities

Matching Pennies

- Goeree and Holt (2001) use 50 subjects and let them play the following game exactly once. In parentheses: choice percentages

	Left (48)	*Right* (52)
Top (**48**)	80, 40	40, 80
Bottom (**52**)	40, 80	80, 40

Matching Pennies

- Goeree and Holt (2001) also let their subjects play two modified versions of the matching pennies game:

	Left (16)	*Right* (84)
Top (**96**)	320, 40	40, 80
Bottom (**4**)	40, 80	80, 40

	Left (80)	*Right* (20)
Top (**8**)	44, 40	40, 80
Bottom (**92**)	40, 80	80, 40

- NE for the first game:

Matching Pennies

- Goeree and Holt (2001) also let their subjects play two modified versions of the matching pennies game:

	Left (16)	*Right* (84)
Top (**96**)	320, 40	40, 80
Bottom (**4**)	40, 80	80, 40

	Left (80)	*Right* (20)
Top (**8**)	44, 40	40, 80
Bottom (**92**)	40, 80	80, 40

- NE for the first game: p(up)=0.5, p(left)=1/8

Matching Pennies

- Goeree and Holt (2001) also let their subjects play two modified versions of the matching pennies game:

	Left (16)	*Right* (84)
Top (**96**)	320, 40	40, 80
Bottom (**4**)	40, 80	80, 40

	Left (80)	*Right* (20)
Top (**8**)	44, 40	40, 80
Bottom (**92**)	40, 80	80, 40

- NE for the first game: p(up)=0.5, p(left)=1/8
- NE for the second game:

Matching Pennies

- Goeree and Holt (2001) also let their subjects play two modified versions of the matching pennies game:

	Left (16)	*Right* (84)
Top (**96**)	320, 40	40, 80
Bottom (**4**)	40, 80	80, 40

	Left (80)	*Right* (20)
Top (**8**)	44, 40	40, 80
Bottom (**92**)	40, 80	80, 40

- NE for the first game: p(up)=0.5, p(left)=1/8
- NE for the second game: p(up)=0.5, p(left)=10/11

Logit QRE

- We saw that NE does not always predict behavior well
- Two possible modifications
  - Relax the assumption that beliefs are consistent
  - Relax the assumption that agents best respond given beliefs

Logit QRE

- We saw that NE does not always predict behavior well
- Two possible modifications
  - Relax the assumption that beliefs are consistent
  - Relax the assumption that agents best respond given beliefs
- Logit Quantal Response Equilibrium (QRE) relaxes the second assumption
- Agents no longer always select the strategy with the highest expected payoff...
- ...but they are more likely to play strategies with high expected payoffs as opposed to strategies with low expected payoffs
- Beliefs are still consistent: agents are aware of the fact that others make mistakes when they compute expected payoffs

Logit QRE

- n players
- Player i can choose among J_i pure strategies
- $\pi_{ij}(\sigma)$ = Expected payoff for player i when i plays pure strategy s_{ij} and others follow the mixed strategy profile σ_{-i}
- $\hat{\pi}_{ij} = \pi_{ij}(\sigma) + \epsilon_{ij}$
- $\epsilon_i = (\epsilon_{i1}, \ldots, \epsilon_{iJ_i})$, with pdf $f_i(\epsilon_i)$
- In logit QRE, we assume $f_i(\epsilon_i)$ follows an extreme value distribution and the ϵ_{ij} are iid
- It then follows that $\sigma_{ij}(\pi_i) = \frac{\exp(\lambda \pi_{ij})}{\sum_{k \in S_i} \exp(\lambda \pi_{ik})}$
  - σ_{ij} is the probability that player i plays his j-th pure strategy

- $ij-$response set:

$$R_{ij}(\pi_i) = \left\{ \varepsilon_i \in \mathbb{R}^{j_i} : \; \pi_{ij}(\sigma) + \varepsilon_{ij} \geq \pi_{ik}(\sigma) + \varepsilon_{ik}, \forall k \right\},$$

i.e. $R_{ij}(\pi_i)$ specifies the region of errors that will lead i to choose action j.

- $ij-$response set:

$$R_{ij}(\pi_i) = \left\{ \varepsilon_i \in \mathbb{R}^{j_i} : \; \pi_{ij}(\sigma) + \varepsilon_{ij} \geq \pi_{ik}(\sigma) + \varepsilon_{ik}, \forall k \right\},$$

  i.e. $R_{ij}(\pi_i)$ specifies the region of errors that will lead i to choose action j.

- probability that player i will choose action j

$$\sigma_{ij}(\pi_i) = \int_{R_{ij}(\pi_i)} f(\varepsilon_i) d\varepsilon_i$$

- $ij-$response set:

$$R_{ij}(\pi_i) = \left\{ \varepsilon_i \in \mathbb{R}^{j_i} : \ \pi_{ij}(\sigma) + \varepsilon_{ij} \geq \pi_{ik}(\sigma) + \varepsilon_{ik}, \forall k \right\},$$

  i.e. $R_{ij}(\pi_i)$ specifies the region of errors that will lead i to choose action j.

- probability that player i will choose action j

$$\sigma_{ij}(\pi_i) = \int_{R_{ij}(\pi_i)} f(\varepsilon_i) d\varepsilon_i$$

- In logit QRE, we assume $f_i(\varepsilon_i)$ follows an extreme value distribution and the ε_{ij} are iid

- $ij-$response set:

$$R_{ij}(\pi_i) = \left\{ \varepsilon_i \in \mathbb{R}^{j_i} : \ \pi_{ij}(\sigma) + \varepsilon_{ij} \geq \pi_{ik}(\sigma) + \varepsilon_{ik}, \forall k \right\},$$

i.e. $R_{ij}(\pi_i)$ specifies the region of errors that will lead i to choose action j.

- probability that player i will choose action j

$$\sigma_{ij}(\pi_i) = \int_{R_{ij}(\pi_i)} f(\varepsilon_i) d\varepsilon_i$$

- In logit QRE, we assume $f_i(\varepsilon_i)$ follows an extreme value distribution and the ε_{ij} are iid
- It then follows that

$$\sigma_{ij}(\pi_i) = \frac{\exp(\lambda \pi_{ij})}{\sum_{k \in S_i} \exp(\lambda \pi_{ik})}$$

Logit QRE

- A mixed strategy profile σ is a logit QRE if the profile of expected payoffs π induced by σ induces σ given *f*
- In other words: Agents have consistent expectations (they know the distribution of choices of other agents)

Properties of Logit QRE

- σ_{ij} is strictly monotonic in π_{ij}
  - Strategies with higher payoffs are chosen more often than strategies with lower payoffs
- When $\lambda \to 0$, all strategies are chosen with equal probability
- When $\lambda \to \infty$, the logit QRE converges to a NE
  - Therefore, logit QRE can also be used as an equilibrium selection device
- What happens when we multiply all payoffs by a constant?

Quantal Response Equilibrium

- Existence: of a QRE (Theorem 1, McKelvey and Palfrey,1995)
  - Vector of player *i*'s error terms: $\epsilon_i = (\epsilon_{i1}, \ldots, \epsilon_{iJ_i})$ is distributed according to joint density $f(\epsilon_i)$
  - Assume the marginal distribution exists for each ϵ_{ij} and $E(\epsilon_i) = 0$
  - A QRE then always exists for normal form games with a finite number of players with a finite number of pure strategies

Properties of Logit QRE

- σ_{ij} is strictly monotonic in π_{ij}
  - Strategies with higher payoffs are chosen more often than strategies with lower payoffs
- When $\lambda \to 0$, all strategies are chosen with equal probability
- When $\lambda \to \infty$, the logit QRE converges to a NE
  - Therefore, logit QRE can also be used as an equilibrium selection device
- What happens when we multiply all payoffs by a constant?
  - Typically changes the logit QRE: The higher expected payoffs relative to the error term, the more likely agents pick the strategy with the highest expected payoff.

Interpretation of error terms

- Error terms are factors only observable to the agent himself while the distribution is common knowledge
  - Distractions, Miscalculations, Misperceptions
    - but: Mistakes are not always related to payoff differences. For example: agents are unable to solve difficult math problems no matter how high the payoff
  - Heterogeneous preferences (spite, envy, altruism)
  - Error terms reflect control costs. These are minimized by uniform randomization. More discriminating strategies are costlier (Mattson and Weibull, 2002)
- If you are unwilling to commit to a specific distribution of error terms: Goeree et al., 2005: regular QRE

Explaining the Centipede Game using logit-QRE

- Suppose $\lambda = 2$
- What is the take probability at the last node?

Explaining the Centipede Game using logit-QRE

- Suppose $\lambda = 2$
- What is the take probability at the last node?
  - We have $\sigma_{ij}(\pi_i) = \frac{\exp(\lambda \pi_{ij})}{\sum_{k \in S_i} \exp(\lambda \pi_{ik})}$
  - Here, the payoff of taking is 2.82 while the payoff of passing is 0.28. Therefore, the probability of taking in a logit QRE is $\frac{\exp(2 \cdot 2.82)}{\exp(2 \cdot 2.82) + \exp(2 \cdot 0.28)} = 0.994$
- What is the take probability at the second to last node?
  - The payoff of taking is 2.69, the expected payoff of passing is $0.994 \cdot 0.38 + 0.006 \cdot 2.92 = 0.396$
  - Therefore, the probability of taking is $\frac{\exp(2 \cdot 2.69)}{\exp(2 \cdot 2.69) + \exp(2 \cdot 0.396)} = 0.989$
- Note: the probability of taking increases with later nodes in the game

Explaining the Centipede Game using logit-QRE

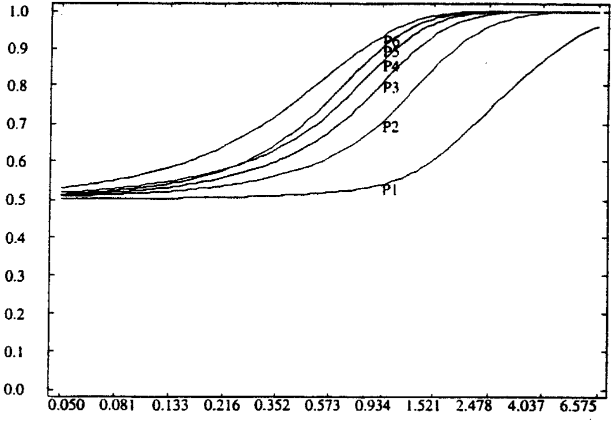


Fig. 5. Quantal response equilibrium of the six-move constant-sum centipede game

- x-axis: λ, y-axis: p(Take)

Centipede Game: Results

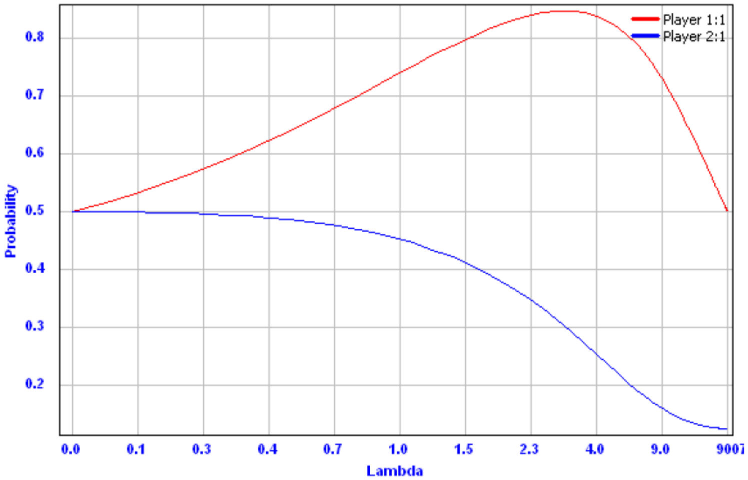
- We also observe
  - that the take probability increases with later nodes in the game
  - that the take probability increases as subjects gain experience
- Logit-QRE accounts for both of these observations

Matching Pennies and Logit QRE

- Consider again the first asymmetric matching pennies game in Goeree and Holt (2001)
- In logit QRE, we have:
  - $p^* = \frac{\exp(\lambda \cdot \pi_{up}(q^*))}{\exp(\lambda \cdot \pi_{up}(q^*)) + \exp(\lambda \cdot \pi_{down}(q^*))}$
  - $p^* \in [0, 1]$
  - $q^* = \frac{\exp(\lambda \cdot \pi_{left}(p^*))}{\exp(\lambda \cdot \pi_{left}(p^*)) + \exp(\lambda \cdot \pi_{right}(p^*))}$
  - $q^* \in [0, 1]$
  - There is rarely an explicit solution to such problems but you can find a fixed point numerically.
    - http://gambit.sourceforge.net
    - Solver in Excel

Matching Matching Pennies and Logit-QRE

- x-axis: λ, y-axis: red: p(up), blue: p(left)



Matching Pennies

	Left (16)	*Right* (84)
Top (**96**)	320, 40	40, 80
Bottom (**4**)	40, 80	80, 40

	Left (80)	*Right* (20)
Top (**8**)	44, 40	40, 80
Bottom (**92**)	40, 80	80, 40

- Compare to the first asymmetric matching pennies game

Matching Pennies and Logit-QRE

Can logit-QRE “explain” what we observe in the experiment?”

- Symmetric matching pennies: The QRE prediction coincides with NE, which fits the data
- In the asymmetric versions of the game, we need to be able to accommodate the *own-payoff effect*
- Logit-QRE indeed predicts an own-payoff effect for *any* value of λ (you can use the monotonicity property to show that)

Matching Pennies and Logit-QRE

- While Logit-QRE does predict the own-payoff effect, the predicted choice probabilities do not fit the data for any level of λ
- Estimates for λ exhibit quite a large variation, which makes it hard to commit to a specific value ex ante
- Other factors are important and can be included in a logit-QRE (e.g., risk/loss aversion)

Matching Pennies and Logit-QRE

- What is the set of logit-QRE in the symmetric matching pennies game studied by Goeree and Holt (2001)?

	Left (48)	*Right* (52)
Top (**48**)	80, 40	40, 80
Bottom (**52**)	40, 80	80, 40

Matching Pennies and Logit-QRE

- Claim: in any logit-QRE, we have $p(up) = p(left) = 0.5$
  - Suppose by means of contradiction that $p(up) > 0.5$
  - We then have $\pi_{right} > \pi_{left}$
  - By strict monotonicity, we therefore have $p(right) > p(left)$
  - Therefore, $\pi_{down} > \pi_{up}$
  - Therefore, by strict monotonicity, $p(down) > p(up)$, which is a contradiction
  - We can use a similar argument to show that $p(up) < 0.5$ generates a contradiction. Therefore, $p(up) = 0.5$ in any logit-QRE. Similarly, we can show that $p(left) = 0.5$ in any logit-QRE.
- Intuitively, the logit-QRE when $\lambda = 0$ is $p(up) = p(left) = 0.5$. This happens to coincide with the mixed NE. Since logit-QRE converges to the NE, we end up with one single logit-QRE.

Matching Pennies and Logit-QRE

- Consider the following asymmetric matching pennies game. Suppose somebody claims that $p(up) = 0.4, p(left) = 0.3$ is a logit-QRE of this game. True or false? Hint: use the monotonicity property of logit-QRE!

	L	R
U	4,0	0,1
D	0,1	1,0

Matching Pennies and Logit-QRE

- Notation
  - p: probability row plays up
  - q: probability column plays left
- Expected Payoffs
  - Row player when playing up: $E(up) = 4q$
  - Row player when playing down: $E(down) = 1 - q$
- Therefore,
  $E(up) > E(down) \Leftrightarrow 4q > 1 - q \Leftrightarrow 5q > 1 \Leftrightarrow q > 0.2$
- Therefore, $p > 0.5 \Leftrightarrow q > 0.2$ and $p < 0.5 \Leftrightarrow q < 0.2$
- Therefore, when $q = 0.3 > 0.2$, we must have $p > 0.5$
- Therefore, $p = 0.4, q = 0.3$ cannot be a logit-QRE for any value of λ
- This is also true for a wide range of other distributions of ϵ as long as monotonicity is satisfied

Discussion

Challenges

- heterogeneity
  - accomodate by estimating a distribution of λ
  - combine logit QRE with other models
- can explain a large set of outcomes
  - λ is not stable across games. may not be able to settle for a specific value
  - account for other factors (e.g., difficulty to compute expected payoffs)
- does it really describe how people decide?
  - costly mistakes are not always less frequent than cheap ones (example: students writing an exam)
  - but: fairly robust
- and we do need some tremble, this one seems better than many alternatives