OPTIMISM AND PESSIMISM IN GAMES*

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Abstract

This paper considers the impact of ambiguity in strategic situations. It chiefly extends the earlier literature by allowing for optimistic responses to ambiguity. In additions it extends the previous literature to a larger class of games. For instance neither symmetry nor concavity in own strategy is assumed. We use the CEU model of ambiguity. A new solution concept for players who may express ambiguity-preference is proposed. Then we study comparative statics of changes in ambiguity-attitude in games with strategic complements. This gives a precise statement of the impact of ambiguity on economic behaviour.

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1 INTRODUCTION

1.1 Background

This paper considers the impact of ambiguity on systems of agents who interact in the presence of strategic complementarity. There are many examples in economics, in particular it can be seen as a stylized representation of financial contagion.

Ambiguity describes situations where individuals cannot or do not assign subjective probabilities to uncertain events. In contrast we shall use *risk* to refer to situations where the decision-maker is familiar with the relevant probabilities. Theoretical models of the impact of ambiguity on individual decisions can be found in Gilboa and Schmeidler [27], Sarin and Wakker [47] or Schmeidler [49]. In Eichberger and Kelsey [15] we studied games of strategic complements or substitutes where players were ambiguity-averse.[1] In particular we showed that in games of strategic complements, the comparative statics of ambiguity made testable predictions. In games of positive externalities and strategic complements, an increase in ambiguity-aversion has the effect of decreasing equilibrium strategies.[2] A possible criticism of the previous literature is that experimental evidence shows individuals are not uniformly ambiguity-averse.[3] While ambiguity-aversion is common, individuals do at times display ambiguity-preference. The present paper aims to study the case where individuals may (but are not required to) express ambiguity-preference.

There is a substantial body of experimental evidence which suggests that people behave differently in the presence of ambiguity than in situations where probabilities are well defined, (Camerer and Weber [4]). The importance of the distinction between risk and ambiguity is confirmed by research, which shows that different parts of the brain process ambiguity and probabilistic risk, see Camerer, Lowenstein, and Prelec [5]. The majority of individuals respond by behaving cautiously when there is ambiguity. Henceforth we shall refer to such cautious behaviour as *ambiguity-aversion.* In experiments a minority of individuals behave in the opposite way which we shall refer to as *ambiguity-preference,* (Kilka and Weber [35]). Moreover the same individual may express both ambiguity-preference and ambiguity aversion in different contexts.

[1] This built on an earlier literature on games with ambiguity initiated by Dow and Werlang [13].

[2] In this paper we considered games with an ordering on the strategy space. There is strategic complementarity if when one player increases his/her strategy this gives other players an incentive to increase their strategies.

[3] One exception is Marinacci [39], who assumes that players either display global ambiguity-aversion or global ambiguity-preference. However the evidence shows that the same individual can express ambiguity-preference in some situations and ambiguity-aversion in others. We consider such mixed ambiguity attitudes.

Experimental evidence shows that in situations of unknown probabilities there is neither uniform ambiguity-aversion nor uniform ambiguity-preference. Rather the subjective weights attached to events has an inverse-S shape.[4] This implies that there is ambiguity-seeking for relatively unlikely events and ambiguity-aversion for more likely events. Abdellaoui, Vossmann, and Weber [1], Kilka and Weber [35] and Wu and Gonzalez [56] find experimental evidence which supports this hypothesis.

1.2 Ambiguity in Games

In this paper we examine the impact of ambiguity in games with positive externalities and increasing differences. Ambiguity is modelled by Choquet expected utility (henceforth CEU), Schmeidler [49]. The paper extends the previous literature on ambiguity and strategic interaction by considering both a larger class of games and a larger class of preferences than the extant literature. In particular we allow for optimistic responses to ambiguity. As we shall argue, previous notions of equilibrium and the support of a capacity have explicitly or implicitly assumed ambiguity-aversion. Hence they may not be suitable for situations in which players may express ambiguity-preference. We propose a new definition of support, which we believe is more appropriate, and use it as the basis of an equilibrium concept for games with ambiguity.

This solution concept allows us to study the comparative statics of ambiguity-attitude. We find that an increase in optimism has the effect of increasing the equilibrium strategy. If a given player is optimistic, (s)he places more weight on good outcomes than an expected utility maximiser would. In this case, good outcomes would be perceived to be situations where other players use high strategies. Increasing differences is a form of strategic complementarity, which implies that over-weighting high strategies will increase the given player's incentive to play a higher strategy. Thus the combination of an increase in optimism and increasing differences will increase the best response of any given individual and hence the equilibrium strategy.

Strategic complementarity can lead to multiple equilibria. In this case we can show that if there is sufficient ambiguity, equilibrium will be unique. If agents are sufficiently optimistic (resp. pessimistic) this equilibrium will be higher (resp. lower) than the highest (resp. lowest) equilibria without ambiguity. Note that ambiguity and ambiguity-attitude have distinct

[4] This comment is heuristic rather than precise. It is only possible to represent the decision-weights graphically if one restricts attention to a one-dimensional sub-family of events.

effects. Ambiguity causes the set of equilibria to collapse to a single equilibrium, while an increase/decrease) in optimism causes the set of equilibria to move up (down).

1.3 Applications

Allowing for optimism is useful, since it allows us to model phenomena where ambiguity-preference plays an important role in motivating behaviour. This might include setting up small businesses, speculative research and development and decisions to enter careers such as acting or rock music where the returns are very uncertain. As an example of potential applications we consider the weakest link public goods model.

It has long been suspected that ambiguity plays an important role in financial markets, especially during bank runs, stock market booms and crashes. Keynes [34] spoke of 'waves of optimism and pessimism'. Ambiguity-aversion can be used to model some of these phenomena, for instance the model of bank runs in Eichberger and Spanjers [17]. However it seems clear that we need to allow for ambiguity-loving behaviour if we wish to model asset bubbles. One could argue that the financial system has multiple equilibria, one with a high level of activity and high asset values and one with a low level of activity and low asset values. Strategic complementarity is present since when stockmarket prices are higher it is easier to use financial assets as collateral. Optimism may have played a part in the asset price inflation which preceded the recent financial collapse. One factor in the credit crunch may have been an increased perception of ambiguity arising from doubts about the value of complex derivative securities. Combined with an increase in pessimism this would have had a negative impact on the financial system. This is not to deny that there are also real causes of the banking crisis. However changes in the perceptions of ambiguity and attitudes to it, may well have played a part in the financial crisis and the boom which preceded it. This may have been amplified by the interaction between ambiguity and strategic complementarity.

Teitelbaum [53] considers ambiguity in the context of tort law, (i.e. law concerning accidents). Typically accidents are rare events. Consequently individuals will have little opportunity to observe relative frequencies. Thus it is not implausible that they may regard accidents as ambiguous. Ambiguity-preference plays an important role in his model. Tort law effectively concerns a game between an injurer and a victim. Both parties may choose a level of care (e.g. how safely they drive) and a level of activity (e.g. how frequently they drive). Uncertainty

about the occurrence of an accident and the damage done if there is an accident, depend on the actions taken by both parties. Teitelbaum argues that ambiguity-preference may result in individuals taking too little care to avoid an accident under conventional liability rules such as negligence.

Ambiguity preference is also useful for explaining experimental results. Experimental evidence rarely finds behaviour which is uniformly ambiguity-averse. For instance Goeree and Holt [28] have an experimental study of ten games. In each they have a "treasure" treatment for which the evidence strongly supports Nash equilibrium. However they also have a "contradiction treatment" in which an apparently irrelevant parameter is changed. In this treatment, the experimental evidence is quite strikingly inconsistent with Nash predictions. In Eichberger and Kelsey [16] we show that many of these experimental results can be explained by the hypothesis that players view their opponents' behaviour as ambiguous. Optimistic attitudes to ambiguity form an essential part of our explanation. It is clear that in the experiments of Goeree and Holt [28] subjects are over-weighting high as well as low outcomes. This can be explained by ambiguity if one allows for the possibility of ambiguity-preference.

Organization of the Paper In section 2 we present our framework and definitions. In section 3 we introduce our solution concept and prove existence of equilibrium. In section 4 we derive the comparative statics of changes of ambiguity-attitude in games of strategic complements. An application to the weakest link public goods model is discussed in section 5 and concluding comments are in Section 7. Appendix A relates a number of alternative notions of the support of a capacity, some examples of equilibrium under ambiguity can be found in Appendix B and Appendix C contains the proofs of those results not proved in the text.

2 MODELLING AMBIGUITY IN GAMES

We consider a game $\Gamma = \langle N; (S_i), (u_i) : 1 \leqslant i \leqslant n \rangle$ with finite pure strategy sets S_i for each player and payoff functions $u_i(s_i, s_{-i})$. The notation, s_{-i}, indicates a strategy combination for all players except i. The space of all strategy profiles for i's opponents is denoted by S_{-i}. The space of all strategy profiles is denoted by S. Player i has utility function $u_i : S \to \mathbb{R}$, for $i = 1, ..., n$.

We want to model ambiguity about the possible behaviour of a player's opponents. For

ambiguity-averse players, Choquet Expected Utility (henceforth CEU) provides a suitable representation for choice under ambiguity. In this case, Schmeidler [49] proves that CEU also has a multiple-prior representation. The multiple prior model, also know as maxmin expected utility (MEU), was axiomatized by Gilboa and Schmeidler [27]. This property allows us to interpret CEU as ignorance about the true probability distribution.

In the following subsections, we relax the assumption of ambiguity-aversion. For a certain class of beliefs, the CEU model coincides with the α-multiple prior expected utility model (α-MEU). Though there is no behavioural axiomatization of the latter model, it offers a natural distinction between ambiguity and ambiguity-attitude, be it optimistic or pessimistic.[5]

2.1 Non-Additive Beliefs and Choquet Integrals

The CEU model of ambiguity represents beliefs as capacities. A capacity assigns non-additive weights to subsets of S_{-i}. Formally, they are defined as follows.

Definition 2.1 *A* capacity *on S_{-i} is a real-valued function ν on the subsets of S_{-i} such that $A \subseteq B \Rightarrow \nu(A) \leqslant \nu(B)$ and $\nu(\varnothing) = 0,\ \nu(S_{-i}) = 1$.*

Thus a capacity is like a subjective probability except that it may be non-additive.

If beliefs are represented by a capacity ν_i on S_{-i}, the expected utility of the payoff obtained from a given act, can be found using the Choquet integral, which is defined below.

Definition 2.2 *The* Choquet integral *of $u_i(s_i, s_{-i})$ with respect to capacity ν on S_{-i} is:*

$$V_i(s_i) = \int u_i(s_i, s_{-i})\, d\nu = u_i\left(s_i, s_{-i}^1\right)\nu\left(s_{-i}^1\right) + \sum_{r=2}^{R} u_i\left(s_i, s_{-i}^r\right)\left[\nu\left(s_{-i}^1, ..., s_{-i}^r\right) - \nu\left(s_{-i}^1, ..., s_{-i}^{r-1}\right)\right],$$

where the strategy profiles in S_{-i} are numbered so that $u_i\left(s_i, s_{-i}^1\right) \geqslant u_i\left(s_i, s_{-i}^2\right) \geqslant ... \geqslant u_i\left(s_i, s_{-i}^R\right)$.

A simple, though extreme, example of a capacity is the complete uncertainty capacity defined below.

[5] For the special case, where ambiguity is restricted to the categories of certainty, possibility, and impossibility, Chateauneuf, Eichberger, and Grant [6] provide an axiomatization in the Savage framework. In Ghirardato, Maccheroni, and Marinacci [26] a sub-class of α-MEU preferences over infinite state spaces is axiomatized. See also Eichberger, Grant, Kelsey, and Koshevoy [20] for an example of preferences which satisfy their axioms.

Example 2.1 *The* complete uncertainty capacity, ν_0 *on* S_{-i} *is defined by* $\nu_0(S_{-i}) = 1, = \nu_0(A) = 0$ *for all* $A \subsetneqq S_{-i}$.

Intuitively ν_0 describes a situation where the decision maker knows which states are possible but has no further information about their likelihood. At the other extreme is an additive probability distribution which satisfies $\nu(A \cup B) = \nu(A) + \nu(B) - \nu(A \cap B)$ for any $A, B \subseteq S$. One can view complete uncertainty as describing the case where there is the greatest possible ambiguity. In contrast, an additive probability describes a situation in which the true probabilities are known with certainty, i.e. there is no ambiguity. (Alternatively if individuals' subjective beliefs are additive, they behave as if they knew the probabilities without doubt.) Further examples will be provided throughout this paper.

Definition 2.3 *A capacity,* μ, *is said to be* convex *if* $\mu(A \cup B) \geqslant \mu(A) + \mu(B) - \mu(A \cap B)$.

Convex capacities can be associated in a natural way with a set of probability distributions called *core* of the capacity.

Definition 2.4 *Let* μ *be a capacity on* S_{-i}. *The* core, $\mathcal{C}(\mu)$, *is defined by,*

$$\mathcal{C}(\mu) = \{p \in \Delta(S_{-i}); \forall A \subseteq S_{-i}, p(A) \geqslant \nu(A)\}.$$

The core of a convex capacity is always non-empty. Due to ambiguity, a given player may not be able to assign a single probability over his/her opponents' strategy spaces which represents that player's beliefs. Instead (s)he considers a number of probability distributions to be possible. The core is the closed convex hull of this set of probabilities. The capacity ν in Example 2.1 is convex and has the set of all probability distributions as its core, $\mathcal{C}(\mu) = \Delta(S_{-i})$, which explains the name *complete uncertainty capacity.*

For convex capacities, one can interpret the core of the capacity as upper bounds on the probabilities of events. Below we define the *dual capacity* which can be associated with an arbitrary capacity. The capacity and its dual provide alternative representation of the same information.

Definition 2.5 *Let* ν *be a capacity on* S_{-i} *and denote by* $\neg A := S_{-i} \backslash A$ *the complement of the event* A. *The* dual capacity $\overline{\nu}$ *on* S *is defined by* $\overline{\nu}(A) = 1 - \nu(\neg A)$.

The capacity and its dual encode the same information. For a convex capacity μ, any probability distribution p in the core $\mathcal{C}(\mu)$ satisfies:

$$\mu(A) \leqslant \sum_{s_{-i} \in A} p(s_{-i}) \leqslant \overline{\mu}(A).$$

Notice that, for a convex capacity, $\mu(A) \leqslant \overline{\mu}(A) := 1 - \mu(\neg A)$ holds for any $A \subseteq S_{-i}$. If the inequality is an equality for all A in S_{-i}, then μ is a probability distribution. Since a capacity and its dual represent upper and lower bounds for the probability distributions in the core of a convex capacity it is natural to define the degree of ambiguity of a player as follows.

Definition 2.6 *Let μ be a convex capacity on S_{-i}. Define the maximal (resp. minimal) degrees of ambiguity of μ by:*

$$\begin{aligned} \lambda(\mu) &= \max\{\overline{\mu}(\neg A) - \mu(A) : \varnothing \subsetneqq A \subsetneqq S_{-i}\}, \\ \gamma(\mu) &= \min\{\overline{\mu}(\neg A) - \mu(A) : \varnothing \subsetneqq A \subsetneqq S_{-i}\}. \end{aligned}$$

The maximal and minimal degrees of ambiguity provide upper and lower bounds on the amount of ambiguity which the decision-maker perceives. These definitions are adapted from Dow and Werlang [12]. The degrees of ambiguity are measures of the deviation from (binary) additivity. For an additive probability they are equal to zero, while for complete uncertainty (Example 2.1) they are equal to one. These two examples are the extreme cases with the highest and lowest degrees of ambiguity. Convex capacities have degrees of ambiguity between these two cases.

The following result shows that for a convex capacity, the Choquet integral of a pay-off function u_i for a given strategy s_i is equal to the minimum over the core of the expected value over u_i. Hence convex capacities provide an attractive representation of pessimism. When a decision-maker does not know the true probabilities, (s)he considers a set of probabilities to be possible and evaluates any given act by the least favourable of these probabilities.

Proposition 2.1 *(Schmeidler [49]) If μ is an convex capacity on S_{-i}, then*

$$\int u_i(s_i, s_{-i})\, d\mu = \min_{p \in \mathcal{C}(\mu)} \mathbf{E}_p u_i(s_i, s_{-i}),$$

where $\mathbf{E}$ *denotes the expected value of* u_i *with respect to the additive probability* p *on* S_{-i}.

Indeed, Schmeidler [49] argues that convex capacities represent ambiguity-aversion. More recently Wakker [55] has shown that convexity is implied by a generalized version of the Allais paradox. This provides another reason to take convex capacities as a representation of ambiguity and the Choquet expected utility as the pessimistic evaluation of acts given this ambiguity.

2.2 Optimism, Pessimism, and JP-Capacities

Most of the literature on decision making under ambiguity restricts attention to ambiguity-averse decision makers. In particular, most applications to strategic ambiguity in games have used this premise.[6] As argued in the introduction, there is evidence that some individuals respond to ambiguity in an optimistic way. In this paper, we would like to provide an extension of the CEU and multiple prior approaches to games with optimistic as well as pessimistic players.

To achieve this will require us to develop concepts for distinguishing ambiguity from ambiguity-attitudes, which are not readily available in the literature. In particular, to our knowledge there is no axiomatic treatment in the Savage approach which offers a behaviourally based distinction between ambiguity and ambiguity-attitudes. The convex capacity model and the minimum expected utility approach of Gilboa and Schmeidler [27] are derived using "ambiguity-aversion" as an axiom. The CEU approach is applicable to general capacities, but for non-convex capacities, the separation between ambiguity and ambiguity-attitude is not clear. A general capacity combines elements which one can interpret as ambiguity or ambiguity-attitude.

In order to deal with this problem in the spirit of Schmeidler [49], we restrict attention to a class of capacities introduced by Jaffray and Philippe [32] which we will refer to as *JP-capacities.* These were originally proposed in the context of a statistical model with upper and lower probabilities. We believe that they are useful for representing ambiguity in games since they are capable of modelling both optimism and pessimism. Recall that $\overline{\mu}$ denotes the dual capacity of μ.

Definition 2.7 *A capacity* ν *on* S_{-i} *is a* JP-capacity *if there exists a convex capacity* μ *and* $\alpha \in [0,1]$, *such that* $\nu = \alpha\mu + (1-\alpha)\,\overline{\mu}$.

[6] There are, however, a few exceptions including, Eichberger, Kelsey, and Schipper [19], and Marinacci [39].

As in Schmeidler [49], ambiguity is represented by a convex capacity μ and its core. The new capacity proposed in Jaffray and Philippe [32] is a convex combination of the capacity μ and its dual. As the following proposition shows, the CEU of a JP-capacity is a convex combination of the minimum and the maximum expected utility over the set of probabilities in the core of μ.

Proposition 2.2 *(Jaffray and Philippe [32]) The CEU of a utility function u_i with respect to a* JP-capacity $\nu = \alpha\mu + (1-\alpha)\,\overline{\mu}$ *on S_{-i} is:*

$$\int u_i\left(s_i, s_{-i}\right) d\nu\left(s_{-i}\right) = \alpha \min_{p\in\mathcal{C}(\mu)} \mathbf{E}_p u_i\left(s_i, s_{-i}\right) + (1-\alpha) \max_{p\in\mathcal{C}(\mu)} \mathbf{E}_p u_i\left(s_i, s_{-i}\right).$$

This result suggests an interpretation of the parameter α as a degree of pessimism, as it gives a weight to the worst expected utility the player could expect from the strategy s_i. If $\alpha = 1$, then we obtain the MEU model axiomatized by Gilboa and Schmeidler [27]. On the other hand, the weight $(1-\alpha)$ given to the best expected utility which a player can obtain with the strategy s_i provides a natural measure for the optimism of a player. For $\alpha = 0$ we deal with a pure optimist, while in general for $\alpha \in (0,1)$, the player's preferences have both optimistic and pessimistic features.

If preferences can be represented as a Choquet integral with respect to a JP capacity then they lie in the intersection of the CEU and α-MEU models. The α-MEU model was formally defined by Marinacci [40]. If beliefs may be represented by JP-capacities perceived ambiguity is represented by the capacity μ, while ambiguity-attitude is represented by α. Hence JP capacities allow a distinction between ambiguity and ambiguity-attitude, which is formalized in the following definitions.[7]

Definition 2.8 *Let ν and ν' be two capacities on S. We say that ν is* more pessimistic than ν' *if for all $A \subseteq S, \nu(A) \leqslant \nu'(A)$.*

The following result shows that for JP-capacities, an increase in α implies an increase in pessimism.

Proposition 2.3 *Suppose that $\hat{\alpha} \geqslant \tilde{\alpha}$ and μ is convex then $\hat{\nu} = \hat{\alpha}\mu + (1-\hat{\alpha})\,\overline{\mu}$ is more pessimistic than $\tilde{\nu} = \tilde{\alpha}\mu + (1-\tilde{\alpha})\,\overline{\mu}$.*

[7] Ghirardato, Maccheroni, and Marinacci [26] present an alternative way to separate ambiguity and ambiguity-attitude. However as we show in Eichberger, Grant, Kelsey, and Koshevoy [20] there are problems with this approach, when the state space is finite. The present paper only considers games with finite strategy sets, which is the equivalent of a finite state space in their framework.

Proof. The result follows from noting that since μ is convex for all $A \subseteq S$, $\mu(A) \leqslant \overline{\mu}(A)$. ■

A useful special case of JP-capacities is the neo-additive capacity, defined in Example 2.2, which generates CEU preferences that display both optimism and pessimism.[8]

Example 2.2 *Let α, δ be real numbers such that $0 < \delta < 1, 0 < \alpha < 1$, define a neo-additive-capacity ν on S_{-i} by*

$$\nu(A) = \delta(1-\alpha) + (1-\delta)\pi(A),$$

for $\emptyset \subsetneqq A \subsetneqq S_{-i}$, where π is an additive probability distribution on S_{-i}.[9]

The neo-additive capacity describes a situation where the decision maker's 'beliefs' are represented by the probability distribution π. However (s)he has some doubts about these beliefs. This ambiguity about the true probability distribution is reflected by the parameter δ. The highest possible level of ambiguity corresponds to $\delta = 1$, while $\delta = 0$ corresponds to no ambiguity. The reaction to these doubts is in part pessimistic and in part optimistic. The parameter α may be interpreted as a measure of ambiguity attitude. This is similar to the interpretation of the corresponding parameter in the definition of a JP-capacity. Hence, neo-additive preferences maintain a separation between ambiguity and ambiguity-attitude, which are measured by δ and α respectively.

The Choquet expected value of a pay-off function $u_i(s_i, \cdot)$ with respect to the neo-additive-capacity ν is given by:

$$\int u_i(s_i, s_{-i})\, d\nu(s_{-i}) = \delta\alpha \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) + \delta(1-\alpha) \max_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) + (1-\delta) \cdot \mathbf{E}_\pi u_i(s_i, s_{-i}).$$

This expression is a weighted averaged of the highest payoff, the lowest payoff and an average payoff. The response to ambiguity is partly optimistic represented by the weight given to the best outcome and partly pessimistic. A neo-additive capacity can be also viewed as a special case of a JP-capacity (see Proposition 3.2). In Chateauneuf, Eichberger, and Grant [6] it is shown

[8] Neo-additive is an abbreviation for non-extremal outcome additive. Neo-additive capacities are axiomatized in Chateauneuf, Eichberger, and Grant [6].

[9] In Chateauneuf, Eichberger, and Grant [6] the neo additive capacity is written in the form $\nu(A) = \delta\alpha + (1-\delta)\pi(A)$. In the main text we have modified the definition of a neo-additive capacity to be consistent with the definition of a JP capacity. As a result, in both cases α is the weight on the min component in the multiple priors representation of the preferences.

that CEU preferences with neo-additive capacities can also be represented in the following form:

$$\int u_i(s_i, s_{-i})\, d\nu(s_{-i}) = \alpha \min_{p \in \mathcal{P}} \mathbf{E}_p u_i(s_i, s_{-i}) + (1-\alpha) \max_{p \in \mathcal{P}} \mathbf{E}_p u_i(s_i, s_{-i}),$$

where $\mathcal{P} = \{p \in \Delta(S_{-i}) : p \geq (1-\delta)\pi\} := \mathcal{C}(\mu)$, where μ is defined by $\mu(A) = (1-\delta)\pi(A)$, $A \subsetneqq S_{-i}$. Thus, $\mathcal{P}$ is the core of the convex capacity $\mu = (1-\delta)\pi$, i.e., the set of measures 'centred' around a fixed $\pi \in \Delta(S_{-i})$.

Given the unresolved issues surrounding the question of how to distinguish ambiguity of beliefs from ambiguity-attitudes,[10] we restrict attention to JP-capacities where this distinction between ambiguity-attitude, as reflected by the parameter α, and ambiguous beliefs, as represented by the convex part of the capacity μ, appears natural. For economic and game-theoretic applications, this simple parametric separation will be particularly useful.

3 EQUILIBRIUM

3.1 Modelling Ambiguity in Games

The aim of the present paper is to study the impact of ambiguity in games. In contrast to Eichberger and Kelsey [14], where we assumed ambiguity aversion, we are interested also in the effect of ambiguity-preference. For modelling players who may be optimistic in the face of uncertainty, the literature offers three main approaches:

1. the Choquet Expected Utility (CEU) approach pioneered by Schmeidler [50] in combination with the type of capacities (*JP-capacities*) advanced in Jaffray and Philippe [31],

2. the α-MEU model, sometimes also referred to as Multiple-Prior model, introduced by Marinacci [41],

3. the smooth model or KMM-model suggested by Klibanoff, Marinacci, and Mukerji [36].

None of these models has an axiomatization on the basis of preferences over Savage acts, which would allow one to distinguish ambiguity from the agent's attitudes towards ambiguity. In the case of CEU, preferences over acts determine the capacity ν uniquely but not the JP-form $\nu = \alpha\mu + (1-\alpha)\overline{\mu}$, which would achieve such a separation. For the α-MEU model there

[10] For more discussion of this issue compare Epstein [23] and Ghirardato and Marinacci [25].

is no axiomatisation so far.[11] In α-MEU the set of priors is not well defined. Siniscalchi [52] shows that there may be more than one set of priors and more than one α, which represent the same preferences.[12] In the axiomatization of the smooth model, the attitude towards ambiguity reflected in ϕ is determined by a second preference order over second-order acts, hence, not derived from preferences over Savage acts alone. We consider the problem of the axiomatic separation of ambiguity attitude and ambiguity as an unresolved problem and a challenge for future research.

The open question of how to separate attitudes from ambiguity by axioms does not prevent a fruitful application of these decision models in economic and game-theoretic problems. In a specific economic or game-theoretic context, however, the way in which ambiguity and ambiguity attitude is modelled in these three representations may prove to be more or less useful. Both the CEU model with a JP capacity and the α-MEU model represent the attitude towards ambiguity by a single parameter α, the smooth model represents attitudes towards ambiguity by the curvature of the function ϕ. The degree of ambiguity, on the other hand, can be easily identified with the size of the set of priors in the case of α-MEU and the convex part of a JP-capacity in the CEU approach. In the smooth model ambiguity is modelled by the probability distribution over the set off possible probability distributions. Hence, there is no simple way to distinguish the degree of ambiguity from the ambiguity-attitude of the decision maker in the smooth model. This property makes the smooth model less useful for game-theoretic applications[13].

In game theory, one may wish to identify some consistent set of beliefs for the players given the ambiguity of the situation. Such a framework allows one to study comparative static reactions to changes in the players' ambiguity attitude and the perceived ambiguity of the situation. In principle α-MEU could also provide an opportunity to do comparative static analysis in games in this sense. Indeed, the equilibrium notion which we propose in this paper is readily adapted to these preferences. Beyond set inclusion, there is, however, no well established way to do comparative statics with ambiguity measured by the "size of the set of priors". In

[11] Ghirardato, Maccheroni, and Marinacci [26] have proposed a way to define a unique set of priors for the α-MEU model. However as we argue in Eichberger, Grant, Kelsey, and Koshevoy [20] there are some problems with this approach. The set of priors which the authors define does not appear to be completely independent of ambiguity attitude. Moreover it is not possible to have a constant ambiguity-attitude when the state space is finite.

[12] See especially the on-line appendix to Siniscalchi [52].

[13] This is not to say that the smooth model may not be extremely useful in other contexts. In particular, the smooth model allows one to use calculus for comparative static exercises.

contrast, CEU with a JP-capacity offers as we feel a natural way to determine a degree of ambiguity which can be used for comparative static exercises. These considerations lead us to prefer the CEU model of ambiguity as we believe that it is the best suited in a game-theoretic context.

3.2 Equilibrium Concepts

Since the publication of Schmeidler [49], a number of solution concepts for games with ambiguity and ambiguity-averse players have been proposed, see for instance Dow and Werlang [13], Lo [37], Marinacci [39] and Ryan [46]. In all of these, the support of a player's beliefs is used to represent the set of strategies that (s)he believes his/her opponents will play. An equilibrium is defined to occur when every profile of strategies in the support consists only of best responses. The main difference between the various solution concepts is that they use different support notions.

Most of the literature deals with ambiguity-aversion or pessimism, in which case capacities are convex. The Choquet integral computes the expected value with respect to capacity value differences in a decreasing order.[14] Consider the case where the capacity is convex. The weight on the best outcome is equal to its capacity, which can be seen as a lower bound on its probability. The weight on the second highest outcome is the capacity of this event minus the capacity the highest outcome, etc. Again this is the smallest weight which can be assigned to the second highest outcome given what has already been assigned to the highest outcome. It can be seen that this is a very cautious way of calculating an expectation. Hence convex capacities can be viewed as a representation of ambiguous beliefs together with a pessimistic attitude towards ambiguity. In this case, the support of a convex capacity is the appropriate concept for the set of strategies which players believe their opponents will play. Moreover, we will argue in this section (and prove in Appendix A) that the support notions, which were suggested in the literature so far, will essentially coincide for convex capacities.

We propose to represent the set of strategies that a player believes his/her opponents will play also by the support of a capacity. In general, capacities reflect both ambiguity and ambiguity-attitudes. It is therefore necessary to separate ambiguity-attitudes from the ambiguous beliefs

[14] One can define a dual version of the Choquet integral based on the lower level sets. In this case, a convex capacity would represent optimistic attitudes (see, e.g., Denneberg [10]).

component of a capacity in order to find an appropriate support notion. Applying support notions, which work well in the context of convex capacities, to non-convex capacities risks confounding ambiguity-attitude and beliefs. As a result they may suggest that all strategy profiles of the opponents lie in the support. For JP-capacities, which we introduced in the previous section, the convex capacity μ on which a JP-capacity is based provides a consistent representation of beliefs. In the following section we discuss the support of ambiguous beliefs, which is a key concept for defining equilibrium in games.

A second problem needs to be addressed for a satisfactory notion of equilibrium under strategic ambiguity. In game theory, it is common to assume that each player believes that his/her opponents act independently. The notion of an equilibrium in mixed strategies of the standard Nash equilibrium approach builds on the natural notion of "independent beliefs" provided by the unique product of additive measures. It is well known (Lo [38], Hendon, Jacobsen, Sloth, and Tranaes [30]) that there is no equivalent obvious product notion for ambiguous beliefs. Lo [38], therefore, argues strongly for giving up the "independent beliefs" notion in models with ambiguity. We will indicate below how one can model independent beliefs by the Möbius product capacity.[15] The discussion of this issue and the formal definition of an Equilibrium under Ambiguity (EUA) will form the second part of the next section.

CEU-payoff functions of players with optimistic attitudes towards ambiguity will not be quasi-concave. Hence, it is not feasible to prove existence of an equilibrium by Kakutani's fixpoint theorem. There are, however, large and for economic applications important classes of games for which existence theorems can be proved using techniques from lattice theory. In the last part of this section, we show existence of equilibrium for games with strategic complements.

3.3 Support of Ambiguous Beliefs

It is not possible to apply definitions of the support from the literature unmodified since many of them have implicitly assumed ambiguity-aversion. Two definitions have been used for ambiguity-averse or pessimistic players with convex capacities, the *Dow-Werlang (DW) support* (Dow and Werlang [13]) and *Marinacci (M) support* (Marinacci [39]).

The *DW-support* of the capacity ν, $\operatorname{supp}_{DW} \nu$ is a set $E \subseteq S_{-i}$, such that $\nu(S_{-i} \backslash E) = 0$

[15] Technically we need to assume that the convex part of a JP-capacity μ is a Möbius independent product of belief functions defined on the marginals. For a definition of the Möbius independent product and further discussion see Ghirardato [24].

and $\nu(F) > 0$, for all F such that $S_{-i} \backslash E \subsetneqq F$. This definition has the advantage that there always exists a support, however it may not be unique. For example, the capacity of complete uncertainty in Example 2.1, $\{s\}$ is a support for any $s \in S_{-i}$. We shall use $D(\nu)$ to denote the set of all DW-supports of the capacity ν.

Marinacci [39] defines the support of a capacity ν to be the set of states with positive capacity. Formally, the *M-support* of capacity ν, is defined by $\operatorname{supp}_M \nu = \{s \in S_{-i} : \nu(s) > 0\}$. Provided it exists, $\operatorname{supp}_M \nu$ is always unique. However there are capacities for which it is empty. Once again, the complete uncertainty capacity in Example 2.1 can serve as an example. In Appendix A (Proposition A.1) we show that, for convex capacities, $\operatorname{supp}_{DW} \nu = \operatorname{supp}_M \nu$ holds, whenever the DW-support is unique.

Based on knowledge concepts from Morris [43], Ryan [46] discusses several notions of a support for ambiguous beliefs which are represented by multiple priors.[16] Since the Choquet expected utility of a convex capacity can also be represented as a multiple-priors functional, where the set of priors is given by the core of the capacity, these support notions can be applied convex capacities.

For a set $P \subseteq \Delta(S_{-i})$ of multiple priors, Ryan [46] (page 56) defines a *strong support of* P as $\bigcup_{p \in P} \operatorname{supp} p$ and a *weak support of* P as $\bigcap_{p \in P} \operatorname{supp} p$, where $\operatorname{supp} p$ denotes the standard notion of a support for additive probability distributions. The strong support of the set of probability distributions P comprises the strategy combinations of the opponents which have a positive probability under *some* probability distribution in the set of priors P, while the weak support contains all strategy combinations which have a positive probability for *all* probability distributions in P. The strong notion of support will never be empty, but will equal the set S_{-i} if there is at least one probability distribution which has full support. On the other hand, the weak support may well be empty.

We prove in Appendix A that, for convex capacities, the support concepts of Dow and Werlang [13] and Marinacci [39] coincide with the weak support notion of Ryan [46] if the DW-support is unique. We believe that this consistency of the support notions for multiple priors

[16] Ryan [46] discusses these notions in a model where decision makers have lexicographically ordered beliefs. In this context, Ryan [46] introduces the concept of *firm beliefs* which coincides with our support notion for the non-lexicographic versions of the CEU and MEU models. An earlier unpublished paper, Ryan [45], contains a similar discussion in the more familiar context of CEU and MEU. In a recent paper, Epstein and Marinacci [22] study the property of "mutual absolute continuity" of all probability distributions in the set of priors. It is easy to see that for finite state spaces this property holds if and only if strong and weak support coincide.

and capacities is a strong argument for using this support notion for convex capacities.

Definition 3.1 *If μ is a* convex *capacity on S_{-i}, we define the support of μ,* $\operatorname{supp}\mu$, *by*

$$\operatorname{supp}\mu = \bigcap\nolimits_{p\in\mathcal{C}(\mu)} \operatorname{supp} p. \tag{1}$$

Definition 3.1 applies the definition of a weak support from Ryan [46] to the set of probabilities in the core of the convex capacity μ. Though the core of μ is never empty, the intersection of the supports of the probability distributions in it may well be empty. Once again, the capacity of complete uncertainty in Example 2.1 can serve as an example. In this case, the core of the capacity is the set of all probability distributions on S_{-i}, $\Delta(S_{-i})$, but the intersection of the supports of the probability distributions in $\Delta(S_{-i})$ is empty.

Though the support defined in equation (1) is a suitable concept for a convex capacity representing the beliefs of the decision-maker, as argued above, this notion is not necessarily adequate for non-convex capacities which represent ambiguity attitudes as well. As an example consider the neo-additive capacity $\nu = \delta(1-\alpha) + (1-\delta)\pi$ from Example 2.2. For $\alpha < 1$, $\operatorname{supp}_{DW}\nu = \operatorname{supp}_M \nu = S_{-i}$ and $\operatorname{supp}\nu = S_{-i}$ as long as the core of ν is not empty. This is not suitable as a support notion for the neo-additive capacity as it does not make a distinction between those strategies which a given player believes are possible for his/her opponents and others.

The problem lies in the fact that neither of these support notions distinguishes between ambiguous beliefs and the decision makers' attitudes towards this ambiguity. For JP-capacities, which are the class of capacities studied in this paper, we propose a support notion which relates only to the convex part μ.

Definition 3.2 *If $\nu = \alpha\mu + (1-\alpha)\overline{\mu}$ is a JP-capacity on S_{-i}, we define the support of ν,* $\operatorname{supp}_{JP}\nu$, *by* $\operatorname{supp}\nu_{JP} = \operatorname{supp}\mu$.

The following result shows that if a strategy profile in is the support of a JP-capacity ν then when this state is added to any non-trivial subset of S_{-i} the capacity of that subset will increase. This implies that the $\hat{s}$ will always receive positive weight in the Choquet integral no matter which act is being evaluated. We interpret this as showing that the individual believes that $\hat{s}$ is possible in the sense that (s)he always gives it positive weight in his/her decision-making.

Proposition 3.1 *Let $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$ is a JP-capacity on S_{-i}, and suppose that $\hat{s} \in \operatorname{supp}\nu_{JP}$, then for all $A \subsetneqq S_{-i}, \nu(A \cup \hat{s}) > \nu(A)$.*

Proof. Since ν is a JP-capacity we may write $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$ for some convex capacity μ. Take $\hat{s} \in \operatorname{supp}\nu$ and $A \subsetneqq S$, $\hat{s} \notin A$. Let $\hat{p} = \operatorname{argmin}_{p\in\mathcal{C}(\mu)} p(A \cup \hat{s})$. Because μ is convex $\mu(A \cup \hat{s}) - \mu(A) = \hat{p}(A \cup \hat{s}) - \mu(A) = \hat{p}(A) - \mu(A) + \hat{p}(\hat{s}) > 0$, since $\hat{p}(A) \geqslant \mu(A)$ and $\hat{p}(\hat{s}) > 0$ by Proposition A.3.

Let $\tilde{p} = \operatorname{argmin}_{p\in\mathcal{C}(\mu)} p(S\backslash A)$. Then $\tilde{\mu}(A \cup \hat{s}) - \tilde{\mu}(A) = \mu(S\backslash A) - \mu(S\backslash(A \cup \hat{s})) \geqslant \tilde{p}(S\backslash A) - \tilde{p}(S\backslash(A \cup \hat{s})) = 1 - \tilde{p}(A) - [1 - \tilde{p}(A \cup \hat{s})] = \tilde{p}(A) + \tilde{p}(\hat{s}) - \tilde{p}(A) > 0$, since $\tilde{p}(\hat{s}) > 0$. Hence for $A \subsetneqq S, s \notin A, \nu(A \cup \tilde{s}) > \nu(A)$. ■

Neo-additive capacities, introduced in Example 2.2, provide an easy and intuitive example for the support of JP-capacities. For such capacities, where the implied set of priors is a δ-neighbourhood of an additive probability π, we can show the JP-support equals the support of π. This is proved in the following result which also finds the maximal and minimal degrees of ambiguity for a neo-additive capacity.

Proposition 3.2 *Let $\nu = \delta(1-\alpha) + (1-\delta)\pi$ be a neo-additive capacity on S_{-i}. Then:*

1. *ν may be written in the form $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$, where $\mu = (1-\delta)\pi(A) + \delta\nu_0(A)$;*[17]
2. *The maximal and minimal degrees of ambiguity of μ are $\lambda(\mu) = \gamma(\mu) = \delta$ respectively;*
3. $\operatorname{supp}_{JP}\nu = \operatorname{supp}\pi$.

Recall we interpret a neo additive capacity as describing a situation where the decision-maker's beliefs are represented by the additive probability distribution π, however (s)he may lack confidence in this belief. Given this, it seems intuitive that the support of the neo-additive capacity should coincide with the support of π. This provides an argument in favour of our definition of support, since it confirms our intuition in this case.

3.4 Independent Beliefs and Equilibrium under Ambiguity

In analogy to a Nash equilibrium we define an *Equilibrium Under Ambiguity (EUA)* to be a situation where each player maximizes his/her (Choquet) expected utility given his/her ambiguous beliefs about the behaviour of his/her opponents. In addition, beliefs have to be reasonable

[17] Recall ν_0 denotes the complete uncertainty capacity.

in the sense that each player believes that his/her opponents play best responses. We interpret this as implying that the support of any given player's beliefs should not be empty and consist only of best responses of the other players. Let $R_i(\nu_i) = \text{argmax}_{s_i \in S_i} \int u_i(s_i, s_{-i})\, d\nu_i(s_{-i})\}$ denote the best response correspondence of player i given beliefs ν_i.

Definition 3.3 *An n-tuple of capacities $\hat{\nu} = \langle \hat{\nu}_1, ..., \hat{\nu}_n \rangle$ is an* Equilibrium Under Ambiguity *if for all players, $i \in I$,*

$$\varnothing \neq \operatorname{supp} \hat{\nu}_i \subseteq \underset{j \neq i}{\times} R_j(\hat{\nu}_j).$$

If there is a strategy profile $\hat{s} = \langle \hat{s}_1, ...\hat{s}_n \rangle$ such that for each player $\hat{s}_{-i} \in \operatorname{supp} \hat{\nu}_i$, we say that $\hat{s}$ is an equilibrium strategy profile*. Moreover if for each player* $\operatorname{supp} \hat{\nu}_i$ *contains a single strategy profile $\hat{s}_{-i}$ we say that $\hat{s}$ is a* pure equilibrium*, otherwise we say that it is* mixed*.*

In equilibrium, the beliefs of player i are represented by a capacity $\hat{\nu}_i$, whose support consists of strategies that are best responses for the opponents. A player's evaluation of a particular strategy may, in part, depend on strategies of the opponents which do not lie in the support. We interpret these as events a player views as unlikely but which cannot be ruled out. This may reflect some doubts the player may have about the rationality of the opponents or whether (s)he correctly understands the structure of the game.

Players choose pure strategies and do not randomize. A mixed equilibrium cannot be interpreted as a randomization. In a mixed equilibrium some player i will have two or more best responses. The support of other players' beliefs about i's play, will contain some or all of them. Thus an equilibrium, where the support contains multiple strategy profiles, is an equilibrium in beliefs rather than in mixed strategies. If the beliefs in an EUA happen to be additive in a two-player game, then an EUA is a Nash equilibrium.

For games with more than two players, however, an EUA with additive beliefs will not be a Nash equilibrium in general, since the Nash equilibrium concept implies two more properties:

(i) players are assumed to play independent strategies (independent choices) and

(ii) any two players hold the same beliefs regarding the other players' choices (third-party consistency).

These properties follow immediately from the Nash equilibrium requirement that beliefs coincide with the (mixed) strategies actually played by the opponents. The independent choices of mixed strategies define a unique probability distribution on the product space of strategy

sets. Both conditions fail for EUA beliefs. In order to avoid these complications, many papers restrict attention to two-player games.[18]

It is well-known (Hendon, Jacobsen, Sloth, and Tranaes [30], Denneberg [10] p. 53-56) that there are several ways of extending the product of capacities from the Cartesian products of the strategy sets to general subsets of the product space. One popular method is the Möbius product capacity, which uses the fact that belief functions or totally monotone capacities have a unique representation by an additive probability distribution over events.[19]

The Möbius product, however, is well-defined only for belief functions.[20] Hence, it cannot be applied directly to JP-capacities, which are not even convex in general. One possibility to obtain a notion of independent beliefs would restrict the convex part μ of a JP-capacity to be a belief function, apply the Möbius product and use the JP-capacity of the Möbius product of μ as the relevant product capacity.[21]

In the light of these complications, Lo [38] argues convincingly for considering correlation in beliefs as the typical case under ambiguity. Some of the arguments for considering correlated beliefs have been put forward already in the context of additive beliefs by Aumann [2]. If beliefs are ambiguous these arguments gain more force. To illustrate how correlated beliefs affect equilibrium beliefs of an EUA, we will consider an example from Aumann [2], which is also discussed in Lo [38]. The EUA concept implies consistency of beliefs about a third player only if they have unique maximizers. In contrast to Nash equilibrium, where players must hold identical beliefs about the opponent's equilibrium mixed strategy, EUA does not constrain beliefs, as the following example illustrates.

Example 3.1 *(Aumann [2], Example 2.3, p. 69) Consider the following three-player game where Player 1 chooses the row,* $S_1 = \{U, D\}$*, Player 2 the column,* $S_2 = \{L, R\}$*, and Player 3*

[18] Lo [37] and Groes, Jacobsen, Sloth, and Tranaes [29] deal with n-player games. They use, however, a conceptually different notion of equilibrium where players choose mixed strategies and hold non-additive beliefs about the other players' mixed strategy choices.

[19] Bailey, Eichberger, and Kelsey [3] use this notion of a product for an application to public goods games.

[20] Belief functions, introduced by Dempster [9] and Shafer [51], also called *totally monotone capacities*, are a special class of capacities with non-negative Möbius parameters.

[21] There are other notions of a product capacity which do not impose restrictions on the marginal capacities. Denneberg [10] provides a good introduction to product capacities and further references to the related literature.

the matrix, $S_3 = \{X, Y\}$. Pay-offs are given in the following matrices:

Player 2

		L	R
Player 1	U	$0, 8, 0$	$3, 3, 3$
	D	$1, 1, 1$	$0, 0, 0$

X

Player 2

		L	R
Player 1	U	$0, 0, 0$	$3, 3, 3$
	D	$1, 1, 1$	$8, 0, 0$

Y

Player 3

Player 3 will be indifferent about the choices X and Y no matter what beliefs (s)he holds. Any Nash equilibrium of this game is of the form $(D, L, \{X, Y\})$, where $\{X, Y\}$ stands for any mixed strategy of Player 3, and yields pay-offs $(1, 1, 1)$.

Aumann [2] argues that, conditional on X, Player 1 would be justified to play U in response to Player 2 choosing R, and, conditional on Y, Player 2 could optimally play R in reply to U with pay-offs $(3, 3, 3)$. Yet, this could be an equilibrium only if the behaviour of Player 1 and Player 2 would be based on inconsistent beliefs about Player 3's choice. According to Aumann [2], such inconsistent beliefs could arise if players have subjective beliefs about a randomizing device which Player 3 uses. For example, Player 3 may follow the strategy of choosing X if there is sunshine and Y otherwise. If Player 1 believes that sunshine will occur with probability $\frac{1}{4}$ and Player 2 assumes that the sun will shine with probability $\frac{3}{4}$, then their choices of U and R respectively, would be justified.

Both types of behaviour can be EUA depending on the ambiguity-attitude of the players and their degrees of ambiguity regarding the behaviour of their opponents. For parameters $(\alpha_1, \alpha_2, \alpha_3)$ and $(\delta_1, \delta_2, \delta_3)$ satisfying

$$3(1 - \delta_1) - 5\alpha_1\delta_1 \geq 0, \quad 3(1 - \delta_2) - 5\alpha_2\delta_2 \geq 0, \tag{2}$$

the neo-additive capacities $(\widehat{\nu}_1, \widehat{\nu}_2, \widehat{\nu}_3)$ with

$$\widehat{\pi}_1((R, X)) = 1, \quad \widehat{\pi}_2((U, Y)) = 1, \quad \widehat{\pi}_3((U, R)) = 1,$$

are an EUA. If the conditions in equation (2) both fail, there are two pure EUA $(\widetilde{\nu}_1, \widetilde{\nu}_2, \widetilde{\nu}_3)$ with

$$\widetilde{\pi}_1((L,X)) = 1, \quad \widetilde{\pi}_2((D,X)) = 1, \quad \widetilde{\pi}_3((D,L)) = 1,$$

and $(\bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3)$ with

$$\bar{\pi}_1((L,Y)) = 1, \quad \bar{\pi}_2((D,Y)) = 1, \quad \bar{\pi}_3((D,L)) = 1,$$

which correspond to the standard Nash equilibria of this game.

Firstly, notice that the correlated beliefs $\widehat{\pi}_1((R,X)) = 1$ and $\widehat{\pi}_2((U,Y)) = 1$ are consistent with players assuming that their opponents choose an optimal strategy. They disagree, however, about which optimal strategy Player 3 will choose. As Aumann [2] suggests this may be due to differing beliefs about the randomizing device.

If beliefs are ambiguous there is even more room for disagreement about how players will behave, if they do not have unique best replies. Note however that ambiguity-attitude may influence the choice of players. In Example 3.1 strong optimism, i.e., high values of α_i may induce behaviour as in the standard Nash equilibrium of the game.

3.5 Existence of Equilibrium for Games with Increasing Differences

Since preferences are not quasi-convex it is not possible to prove existence using standard fixed point arguments. Instead we use lattice-theoretic techniques, which is why we require that the pay-off functions have increasing (or decreasing) differences. An advantage is that we are able to prove existence of a pure equilibrium. Moreover, in the next section, we will show several comparative static results of changes in ambiguity and ambiguity-attitudes for this class of games.

Given that we consider finite strategy sets, we can identify strategy sets with an interval of the integers, $S_i = \{\underline{s}_i, \underline{s}_i + 1, ..., \bar{s}_i\}$, for $i = 1, ..., n$.[22] The payoff function $u_i(s_i, s_{-i})$ satisfies *increasing (resp. decreasing) differences* in $\langle s_i, s_{-i} \rangle$ if when $\hat{s}_i > \tilde{s}_i$, $u_i(\hat{s}_i, s_{-i}) - u_i(\tilde{s}_i, s_{-i})$ is increasing (resp. decreasing) in s_{-i}. If $u_i(s_i, s_{-i})$ satisfies increasing differences in $\langle s_i, s_{-i} \rangle$ then it also has increasing differences in $\langle s_{-i}, s_i \rangle$.[23] Increasing differences implies that if a given

[22] The crucial part of this assumption is the restriction to a finite strategy set. It would be straightforward to extend the results to a multi-dimensional strategy space.

[23] See Topkis [54], p. 42.

player perceives his/her opponents increase their strategy, then (s)he has an incentive to increase his/her strategy as well. Hence it is a form of strategic complementarity. Bertrand oligopoly with linear demand and constant marginal cost would be an example of a game with increasing differences.

Definition 3.4 *A game,* $\Gamma = \langle N; (S_i), (u_i) : 1 \leqslant i \leqslant n \rangle$, *has* positive externalities and increasing differences *if* $u_i(s_i, s_{-i})$ *is increasing in* s_{-i} *and has increasing differences in* $\langle s_i, s_{-i} \rangle$ *for* $1 \leqslant i \leqslant n$.

Positive externalities and increasing differences will be a maintained hypothesis throughout the rest of the paper, i.e. we shall assume that all games satisfy it. Negative externalities may be defined in an analogous way.

The following existence result is proved in Appendix C. Fix a vector of parameters $\alpha = (\alpha_1, ..., \alpha_n) \in [0,1]^n$ representing the ambiguity-attitudes of the players and maximal and minimal degrees of ambiguity $(\lambda, \gamma) = ((\lambda_1, \gamma_1), ...(\lambda_n, \gamma_n))$, $0 \leqslant \gamma_i \leqslant \lambda_i \leqslant 1$, then there exists an Equilibrium Under Ambiguity where players' beliefs are represented by JP-capacities with parameters α and convex parts μ_i satisfying $\lambda_i(\mu) \leqslant \lambda_i$ and $\gamma_i(\mu) \geqslant \gamma_i$.

Theorem 3.1 *Let* Γ *be a game of positive externalities and increasing differences. Then for any exogenously given n-tuples of ambiguity-attitudes* α, *maximal degrees of ambiguity* λ *and minimal degrees of ambiguity* $\gamma, ; (\gamma \leqslant \lambda)$ *the game* Γ *has a pure equilibrium* $\nu = \langle \nu_1, ...\nu_n \rangle$ *in JP-capacities, where* $\nu_i = \alpha_i \mu_i + (1 - \alpha_i) \mu_i$ *for* $1 \leqslant i \leqslant n$. *The convex capacity* μ_i *has, maximal degree of ambiguity at most* λ_i *and minimal degree of ambiguity at least* γ_i.

4 COMPARATIVE STATICS

In this section we investigate the comparative statics of changes in ambiguity-attitude on equilibria. Conducting comparative statics exercises is difficult because the capacity represents three distinct concepts; the perceived ambiguity, the attitude to that ambiguity and the equilibrium beliefs about the opponents' strategies as represented by the support of the capacity. Moreover these are interrelated. For instance if a player's ambiguity-attitude changes this may cause him/her to play a different strategy. The opponents are likely to respond by changing their

strategies, which would require the given player to change his/her beliefs so as to maintain consistency.

Our aim is to investigate the comparative statics of ambiguity-attitude. To do this we need to vary ambiguity-attitude while holding perceived ambiguity constant. We hold ambiguity constant by placing exogenous bounds on the maximal and minimal degrees of ambiguity. (Our comparative static results do not depend on the values of these bounds despite the fact they are exogenous.)

To do this we strengthen positive externalities to the following assumption.

Definition 4.1 *A game,* Γ, *has* positive aggregate externalities *if* $u_i(s_i, s_{-i}) = u_i(s_i, f_i(s_{-i}))$, *for* $1 \leqslant i \leqslant n$, *where* u_i *is increasing in* f_i *and* $f_i : S_{-i} \to \mathbb{R}$ *is increasing in all arguments.*

This is a separability assumption. It says that a player only cares about a one-dimensional aggregate of his/her opponents' strategies. An example would be a situation of team production, in which the utility of a given team member depends on his/her own labour input and the total input supplied by all other members of the team.[24] Negative aggregate externalities may be defined in an analogous way.

4.1 Increasing Differences

For games of positive aggregate externalities with increasing differences, an increase in ambiguity-aversion decreases equilibrium strategies. Intuitively if a given player becomes more ambiguity-averse (s)he will place more weight on outcomes which are perceived as bad. If there are positive externalities, a bad outcome would be when an opponent plays a low strategy. Since increasing differences is a form of strategic complementarity, if a given player increases the decision-weight on low strategies of his/her opponents this will reduce his/her incentive to play a high strategy.

The following theorem is our comparative static result on games with increasing differences. It shows that an increase in pessimism will reduce the equilibrium strategies in games with positive aggregate externalities and strategic complements. If there are multiple equilibria, the strategies played in the highest and lowest equilibria will decrease. For this result, we assume that the ambiguity-attitude of one player changes, the ambiguity-attitudes of other players and the perceived ambiguity are held constant.

[24] For a more detailed analysis of the impact of ambiguity on team production see Kelsey and Spanjers [33].

Theorem 4.1 *Let* Γ *be a game of* positive aggregate externalities *with increasing differences. Assume that beliefs are represented by JP capacities and let* $\alpha = \langle \alpha_1, ..., \alpha_n \rangle$ *denote the vector of ambiguity attitudes. Let* $\underline{s}(\alpha)$ *(resp.* $\bar{s}(\alpha)$*) denote the lowest (resp. highest) equilibrium strategy profile when the minimal (resp. maximal) degree of ambiguity is* γ *(resp.* λ*). Then* $\underline{s}(\alpha)$ *and* $\bar{s}(\alpha)$ *are decreasing functions of* α*.*

Remark 1 *The comparative statics are reversed in games of negative aggregate externalities; for further details see Eichberger and Kelsey [15].*

4.2 Multiple Equilibria

Strategic Complementarity can give rise to multiple Nash equilibria. Under some assumptions we can show if there are multiple equilibria without ambiguity and there is sufficient optimism (resp. pessimism) equilibrium will be unique and will correspond to the highest (resp. lowest) equilibrium without ambiguity. To prove this we need the following assumption.

Assumption 4.1 *For* $1 \leqslant i \leqslant n, u_i(s_i, \bar{s}_{-i})$ *and* $u_i(s_i, \underline{s}_{-i})$ *have a unique maximizer, i.e.* $\left|\text{argmax}_{s_i \in S_i} u_i(s_i, \bar{s}_{-i})\right| = 1$ *and* $\left|\text{argmax}_{s_i \in S_i} u_i(s_i, \underline{s}_{-i})\right| = 1$.

This assumption is required for technical reasons. If the strategy space were continuous and utility were concave in own strategy, it would be implied by our other assumptions. It says that the gaps in the discrete strategy space do not fall in the wrong place. For pay-off functions which are not concave in own strategy, the assumption is more restrictive. The following result is a corollary of Theorem 4.1 and Lemma C.9. Note that the assumption of aggregate externalities is not used in this result.

Proposition 4.1 *Consider a game,* Γ*, of* positive *externalities with increasing differences which satisfies Assumption 4.1. There exist* $\bar{\alpha}$ *(resp.* $\underline{\alpha}$*),* $0 < \underline{\alpha} \leqslant \bar{\alpha} < 1$*, and* $\bar{\gamma}$ *such that if the minimal degree of ambiguity is* $\gamma(\mu_i) \geqslant \bar{\gamma}$ *and* $\alpha_i \geqslant \bar{\alpha}$*, (resp.* $\leqslant \underline{\alpha}$*) for* $1 \leqslant i \leqslant n$*, equilibrium is unique and is smaller (resp. larger) than the smallest (resp. largest) equilibrium without ambiguity.*

Even when Assumption 4.1 is not satisfied, Lemma C.9 shows that as ambiguity increases the Choquet expected pay-offs tend to $\max_{s_i \in S_i} \left\{\alpha_i u_i(s_i, \underline{s}_{-i}) + (1 - \alpha_i) u_i(s_i, \bar{s}_{-i})\right\}$. Thus the

equilibrium pay-offs will be unique even when the equilibrium strategies are not. In a game with increasing differences and multiple Nash equilibria, increasing ambiguity causes the multiplicity of equilibria to disappear, while increasing ambiguity-aversion causes the equilibrium strategies to decrease. Thus ambiguity and ambiguity-attitude have distinct effects. Combined with sufficient optimism, ambiguity can cause the economy to move to a higher level equilibrium.

5 WEAKEST LINK PUBLIC GOODS

This section applies the preceding analysis to the weakest link model of public goods, (Cornes and Sandler [8]). This concerns private provision of a public good. Each player has to decide how much to contribute towards the production of a public good. The level of provision is equal to the minimum contribution rather than the total of individual contributions as in the standard public goods problem. Possible applications include: a number of countries choosing environmental legislation, if industry can relocate to the jurisdiction with the least restrictions and a number of military commanders defending a border, where the enemy can choose the most favourable point to attack.

There are two goods, a public good Y and a private good X and n individuals, $1 \leqslant i \leqslant n$. Each chooses how much to contribute to the production of a public good. Individual i has utility function $v_i(y, x_i) = y - cx_i$, where y denotes the level of public good provision and x_i denotes his/her contribution to the public good (in terms of private good). The marginal cost of a contribution to the public good, is denoted by c, where $0 < c < 1$. Contributions may only take integer values in the range $\underline{s} \leqslant x_i \leqslant \bar{s}$. Thus each player has a finite set of pure strategies. This assumption enables us to apply the results from section 4. Individuals are assumed to have sufficiently large endowments that they are able to contribute $\bar{s}$. The level of public good provision is given by the production function, $y = \min\{x_1, ..., x_n\}$. Thus the utility function may be written in the form: $u_i(x_i, x_{-i}, c) = \min\{x_1, ..., x_n\} - cx_i$.

If there is no ambiguity, Nash equilibrium is not unique. Any situation where all players make the same contribution x^* for any $x^*, \underline{s} \leqslant x^* \leqslant \bar{s}$ is a Nash equilibrium. Only the equilibrium where all make the highest possible contribution $\bar{s}$ is efficient.

Ambiguity has the following effect in this game. Pessimism will cause players to be concerned that others will not contribute, which will render their own contributions useless. On the other

hand optimistic responses to ambiguity will encourage players to choose high strategies and thus make it less likely that there will be an inefficient Nash equilibrium.

The following proposition describes the properties of the weakest-link public goods model.

Proposition 5.1 *The weakest link public goods model is a game of positive aggregate externalities which satisfies Assumption 4.1. It has increasing differences in* $\langle x_i, x_{-i}\rangle$ *and decreasing differences in* $\langle x_i, c\rangle$.

Proof. Positive aggregate externalities are immediate. To see that there are increasing differences in $\langle x_i, x_{-i}\rangle$, assume that $\hat{x}_{-i} > \tilde{x}_{-i}$. Let $\hat{m} = \min_{j\neq i}\{\hat{x}_j\}$and $\tilde{m} = \min_{j\neq i}\{\tilde{x}_j\}$. Then $u_i(x_i, \hat{x}_{-i}, c) - u_i(x_i, \tilde{x}_{-i}, c) = \min\{x_i, \hat{x}_{-i}\} - \min\{x_i, \tilde{x}_{-i}\} = 0, \underline{s} \leqslant x_i \leqslant \tilde{m}; \quad x_i - \tilde{m}, \tilde{m} < x_i \leqslant \hat{m}; \quad \hat{m} - \tilde{m}, \hat{m} < x_i$; which is increasing in x_i. If $\hat{c} > \tilde{c}$, then $u_i(x_i, x_{-i}, \hat{c}) - u_i(x_i, x_{-i}, \tilde{c}) = x_i(\tilde{c} - \hat{c})$, which is decreasing in x_i. Assumption 4.1 holds because $\left|\text{argmax}_{s_i\in S_i} u_i(s_i, \bar{s}_{-i})\right| = \{\bar{s}\}$ and $\left|\text{argmax}_{s_i\in S_i} u_i\left(s_i, \underline{s}_{-i}\right)\right| = \{\underline{s}\}$. ■

The following result is a corollary of Theorem 4.1 and Proposition 5.1.[25]

Corollary 5.1 *In the weakest link public goods model:*

1. *an increase in optimism will increase equilibrium strategies;*
2. *an increase in the cost of contributions, c, will decrease equilibrium strategies;*
3. *if the degree of ambiguity is sufficiently high equilibrium will be unique.*

Goeree and Holt [28] present an experimental study of the weakest link public goods model. Their experiment was run as a one-shot game, which is likely to increase perceived ambiguity. Despite the multiplicity of Nash equilibria they found that subjects tended to concentrate on particular strategies. Subjects tended to play the highest (resp. lowest) strategy when c is low (resp. high). This is compatible with our theoretical results. Optimistic responses to ambiguity are required to explain why, at times, subjects tend to use the highest strategy. Ambiguity-aversion alone could only explain a bias towards low strategies.

Secondly Goeree and Holt found that an increase in c tends to reduce equilibrium strategies. This is despite the fact that Nash equilibrium predicts that c has no effect on play, (provided c remains in the range $0 < c < 1$). As Corollary 5.1 shows that is compatible with our model of ambiguity.

[25] Part 2 is not strictly speaking a corollary. However it may be established by similar techniques.

6 DISCUSSION

6.1 Previous Literature

Compared to our previous work, e.g. Eichberger and Kelsey [15], this paper makes a number of innovations. Much of the contribution of the present paper involves developing techniques for modelling optimistic responses to ambiguity. As a result the set of preferences used has been extended significantly beyond the convex capacity model mainly used in the extant literature. This extension has made new definitions of support and equilibrium necessary. This has enabled us to distinguish between ambiguity and ambiguity-attitude. In our previous research, both ambiguity and ambiguity-attitude were varied simultaneously in comparative static exercises. (It is hard to avoid this in a model which assumes ambiguity-aversion.)

The present paper also studies a significantly broader class of games than Eichberger and Kelsey [15]. The earlier paper which restricted attention to symmetric games of aggregate externalities where the utility function was concave in own strategy. The present paper does not assume symmetry nor does it assume concavity in own strategy. Aggregate externalities are only assumed for the comparative statics section.

6.2 Experimental Studies

There are a relatively small number of experimental studies of ambiguity in games of which we are aware. Colman and Pulford [7] find evidence of ambiguity in games but do not test any particular theory. Eichberger and Kelsey [15] predicted that ambiguity would have the opposite effect in games of strategic complements and substitutes. Eichberger, Kelsey, and Schipper [18] found evidence to support this prediction. Mauro and Castro [42] experimentally test for the impact of ambiguity in models of voluntary contributions to public goods. These are games of strategic substitutes. Their results show that contributions may be above or below the Nash equilibrium level. They interpret this as due to ambiguity-preference or ambiguity-aversion which is compatible with the results in the present paper.

7 CONCLUSION

In this paper we have studied the impact of ambiguity in games. In particular we have extended previous work by proposing new definitions of support and equilibrium, which allow for ambiguity-loving (optimistic) behaviour. Economic applications would include oligopoly models, public goods and environmental problems. These issues are discussed in Eichberger and Kelsey [15] and Eichberger, Kelsey, and Schipper [19]. Most general comparative statics results involve some form of strategic complementarity. Hence we believe that it will not be possible to prove substantially more general results on the comparative statics of ambiguity in games.

A natural extension would be to consider games with strategic substitutes. This is more difficult since it has not been possible to prove general comparative static results for situations of strategic substitutes. In our previous research, Eichberger and Kelsey [15] and Eichberger, Kelsey, and Schipper [19] we have found results for two-person games and games of aggregate externalities with strategic substitutes. However in both of these cases a game of strategic substitutes can be transformed to one of strategic complements by reinterpreting and/or reordering the strategy spaces.

It is possible, in principle, to extend our results to extensive form games. However this will pose some new conceptual problems. Since players will receive new information during the course of play, it will be necessary to model how information is updated.

APPENDIX

A ALTERNATIVE NOTIONS OF SUPPORT

In Section 3.3 we introduced a support notion for convex capacities, $\operatorname{supp} \mu$, and a related notion for non-convex JP-capacities, $\operatorname{supp}_{JP} v$. There we argued that, for convex capacities, the support notion $\operatorname{supp} \mu$ is the most suitable concept because it coincides with all common definitions of support in the literature. In this appendix we will substantiate these claims with some formal results.

For convex capacities, the CEU and MEU representations coincide. This allows for a natural interpretation of a decision maker's ambiguity in terms of the set of prior distributions. For a

set of priors there are essentially two notions of support possible. One can consider only those states as part of the support which have positive probability in all probability distributions of the set of priors, the *weak support* notion of Ryan [46], or one may define the support as the set of states which get positive probability in some probability distribution of the set of priors, the *strong support.* As we will show in this appendix, almost all notions of support for capacities coincide with the weak support notion of multiple priors if the capacity is convex.

In the context of games states correspond to strategy combinations of the opponents. The weak support appears as the natural choice because it does not require best-reply behaviour against any strategy an opponent may possibly play but only against those which are unquestionably played. The alternative notion of the strong support has been studied by Dow and Werlang [11] and Lo [37]. Their work shows that solution concepts for games based on this definition of support do not result in behaviour which is significantly different to that in a Nash equilibrium. The strong support notion seems, therefore, incompatible with the objective of modelling deviations from Nash equilibrium due to ambiguity.

In the context of CEU, the support notion in Definition 3.1 appears also recommended by the role which the states of the weak support play in the Choquet integral. Sarin and Wakker [48] argue that the decision-maker's beliefs may be deduced from the decision weights used in computing the Choquet integral. Considering those states as candidates for the support which always increase the decision weight of the outcome associated with it, one defines the set of *decision-weight increasing states* of a capacity v,

$$\mathcal{B}(\nu) = \{s \in S_{-i} : \forall A \subsetneq S_{-i}, s \notin A; \nu(A \cup s) > \nu(A)\},$$

as the set of states which increase the capacity value of any event A they are joined with. Put differently, the set $\mathcal{B}(\nu)$ consists of those states which always get positive weight in the Choquet integral, no matter which act is being evaluated.[26]

The following Proposition shows that the notion of support suggested in Definition 3.1 coincides with the M-support and the set of states which always receive positive weight in the Choquet integral $\mathcal{B}(\nu)$. Even the DW-support, which is in general non-unique, is closely related

[26] In an earlier draft we used $\mathcal{B}(\nu)$ as our support notion. We obtained similar results to those reported in the present paper. This suggests that our results are reasonably robust.

to the support of Definition 3.1 as Proposition A.1 and Proposition A.2 demonstrate.[27]

Proposition A.1 *For a* convex *capacity* μ :

1. *if* supp_{DW} *is unique, then* $\operatorname{supp}\mu = \mathcal{B}(\mu) = \operatorname{supp}_M \mu = \operatorname{supp}_{DW}\mu$,

2. *otherwise,* $\operatorname{supp}\mu = \mathcal{B}(\mu) = \operatorname{supp}_M \mu \subseteq \operatorname{supp}_{DW}\mu$.

Proof. The proof of Proposition A.1 consists of three lemmata (below). Part 1 follows from Lemma A.3 and Lemma A.2. Part 2 follows from Lemma A.3 and Lemma A.1. ■

Lemma A.1 *If* ν *is a capacity and if* $\operatorname{supp}_M \nu \neq \varnothing$, *then* $\operatorname{supp}_M \nu \subseteq \operatorname{supp}_{DW}\nu$.

Proof. Let $\tilde{s} \in \operatorname{supp}_M \nu$ and suppose, if possible, $\tilde{s} \notin \operatorname{supp}_{DW}\nu$. Then $0 = \nu(S \backslash \operatorname{supp}_{DW}\nu) \geqslant \nu(\tilde{s}) > 0$, which is a contradiction. ■

Note that for $\operatorname{supp}_M \nu = \varnothing$, the support $\operatorname{supp}_{DW}\nu$ is not unique.

Lemma A.2 *Let* ν *be a capacity on* S_{-i} *then* $\operatorname{supp}_{DW}\nu$ *is unique if and only if* $\operatorname{supp}_M \nu$ *is a DW-support.*

Proof. Suppose that $\operatorname{supp}_{DW}\nu$ is unique. Let E be the DW-support. By Lemma A.1, $\operatorname{supp}_M \nu \subseteq E$. Suppose, if possible, there exists $\hat{s} \in E \backslash \operatorname{supp}_M \nu$, then $\nu(\hat{s}) = 0$. Hence, $F := S_{-i} \backslash \{\hat{s}\}$ satisfies $\nu(S_{-i} \backslash F) = 0$. Let G be a minimal set such that $G \subseteq F$ and $\nu(S_{-i} \backslash G) = 0$. Take $\bar{s} \in G$ and let $G' = G \backslash \{\bar{s}\}$. Then by minimality $\nu(S_{-i} \backslash G') > 0$, which establishes that G is a DW-support different from E. However this contradicts uniqueness. Hence $\operatorname{supp}_M \nu = \operatorname{supp}_{DW}\nu$.

Now suppose $M = \operatorname{supp}_M \nu$ is a DW-support. Let F be an arbitrary DW-support. By Lemma A.1, $\operatorname{supp}_M \nu \subseteq F$. Suppose if possible that there exists $\tilde{s} \in F \backslash M$. Let $F' = F \backslash \{\tilde{s}\}$. Then since $F' \subsetneqq F$ and F is a Dow-Werlang support $\nu(S_{-i} \backslash F') > 0$. However since M is a Dow-Werlang support and $S_{-i} \backslash F' \subseteq S_{-i} \backslash M$ we must have $\nu(S_{-i} \backslash F') = \nu(S_{-i} \backslash M) = 0$, which is a contradiction. The result follows. ■

Lemma A.3 *If* μ *is an convex capacity with support* $\operatorname{supp}\mu$, *then* $\operatorname{supp}\mu = \operatorname{supp}_M \mu = \mathcal{B}(\mu)$.

[27] This is in the in the spirit of Sarin and Wakker [48] who argue that beliefs may be deduced from the decision weights used in the Choquet integral.

Proof. Let $\tilde{s} \in \operatorname{supp}_M \mu$ and let $\pi \in \mathcal{C}(\mu)$.[28] Then, by definition, $\pi(\tilde{s}) \geqslant \mu(\{\tilde{s}\}) > 0$. Hence, $\operatorname{supp}_M \mu \subseteq \bigcap_{\pi \in \mathcal{C}(\mu)} \operatorname{supp} \pi = \operatorname{supp} \mu$. On the other hand, suppose $s \in \bigcap_{\pi \in \mathcal{C}(\mu)} \operatorname{supp} \pi$. Since μ is convex, $\mu(s) = \min_{\pi \in \mathcal{C}(\mu)} \pi(s) > 0$.[29] Hence $\bigcap_{\pi \in \mathcal{C}(\mu)} \operatorname{supp} \pi \subseteq \operatorname{supp}_M \mu$.

Suppose $s \in \operatorname{supp}_M \mu$. Then $\mu(s) > 0$. For any $A \subseteq S_{-i}$, $s \notin A$, by convexity of μ, $\mu(A \cup s) \geqslant \mu(A) + \mu(s) > \mu(A)$. Hence, $s \in \mathcal{B}(\mu)$. Conversely suppose $s \in \mathcal{B}(\mu)$, then $\mu(s) = \mu(\varnothing \cup s) > \mu(\varnothing) = 0$. Hence, $s \in \operatorname{supp}_M \mu$. Thus $\operatorname{supp}_M \mu = \mathcal{B}(\mu)$. The result follows. ■

The relationship between the different support notions becomes more clear if one considers so-called *null-additive* capacities.[30]

Definition A.1 *A capacity ν is called* null-additive *iff $\nu(E \cup F) = \nu(E)$ for all $F \subseteq S_{-i}$ with $F \cap E = \varnothing$, $\nu(F) = 0$ and $E \cup F \neq S_{-i}$.*

Proposition A.2 *Let ν be a null-additive capacity on S_{-i}, then,*

1. $D(\nu) = \{\operatorname{supp}_M \nu\} \Longleftrightarrow \operatorname{supp}_M \nu \neq \varnothing$,
2. $D(\nu) = \{\{s_{-i}\} |\ s_{-i} \in S_{-i}\} \Longleftrightarrow \operatorname{supp}_M \nu = \varnothing$.[31]

Proof. Part 1. $\operatorname{supp}_M \nu \neq \varnothing$.

(a) Assume $\operatorname{supp}_M \nu \neq \varnothing$. By Lemma A.1, $\operatorname{supp}_M \nu \subseteq \operatorname{supp}_{DW} \nu$ for any $\operatorname{supp}_{DW} \nu \in D(\nu)$. In order to show that, in this case, the unique DW-support equals the M-support, i.e., $D(\nu) = \{\operatorname{supp}_M \nu\}$, we argue by contradiction. Let $s \in \operatorname{supp}_{DW} \nu \backslash \operatorname{supp}_M \nu$ for some $\operatorname{supp}_{DW} \nu \in D(\nu)$, then $\nu(s) = 0$. Consider the set $E := (S \backslash \operatorname{supp}_{DW} \nu) \cup \{s\}$. By null-additivity, $\nu(E) = 0$. Since $(S \backslash \operatorname{supp}_{DW} \nu) \subsetneqq E$, this contradicts the definition of a DW support. Hence, $D(\nu) = \{\operatorname{supp}_M \nu\}$.

(b) Assume $D(\nu) = \{\operatorname{supp}_M \nu\}$. Since $\operatorname{supp}_{DW} \nu$ is always non-empty, we have $\operatorname{supp}_M \nu \neq \varnothing$.

Part 2. $\operatorname{supp}_M \nu = \varnothing$.

(a) If $D(\nu) = \{\{s_{-i}\} |\ s_{-i} \in S_{-i}\}$ then $\bigcap_{E \in \mathcal{D}(\nu)} E = \varnothing$, $\bigcup_{E \in \mathcal{D}(\nu)} (S_{-i} \backslash E) = S_{-i}$. Then $\nu(S_{-i} \backslash E) = 0$ for all E implies $\nu(s) = 0$ for all $s \in S_{-i}$, i.e., $\operatorname{supp}_M \nu = \varnothing$.

(b) If $\operatorname{supp}_M \nu = \varnothing$, one has $\nu(s) = 0$ for all $s \in S_{-i}$. Null-additivity implies that $\nu(E) = \nu(\bigcup_{s \in E} \{s\}) = 0$ for all $E \subsetneqq S_{-i}$. In particular, for any $s \in S_{-i}$, $\nu(S_{-i} \backslash s) = 0$ and $F = S_{-i}$ is the unique set with positive capacity which includes $S_{-i} \backslash s$. Hence, $\{s\} \in D(\nu)$ for all $s \in S_{-i}$. ■

[28] Recall $\mathcal{C}(\mu)$ denotes the core of the capacity μ, see Definition 2.4.

[29] Although $\mathcal{C}(\mu)$ is an infinite set, the minimum must occur at one of the extremal points. The set of extremal points of a core is finite. Thus the minimum must be positive.

[30] Pap [44] studies null-additive capacities in detail.

[31] Recall, $\mathcal{D}(\nu)$ denotes the set of DW-supports of the capacity ν.

If a capacity is null-additive, then the Marinacci support is the intersection of the DW-supports and it is empty if and only if there are multiple DW-supports without a common intersection.

Many capacities in economic applications are null-additive, e.g. neo-additive capacities[32]. We do not suggest, however, to impose this condition in general. It does not appear to be unusual for an ambiguous situation that a decision-maker may be able to express a likelihood for an event but not for its sub-events. For example, in the Ellsberg three-colour urn (Ellsberg [21]), where it is known that there are 30 red balls and 60 balls which are either black or yellow, it appears natural to assume that the likelihood of the ball drawn being black or yellow is $\frac{2}{3}$. Yet, it does not follow that one can rule out that there are no black or no yellow balls in the urn. This would imply capacity values, $v(B) = v(Y) = 0$ and $v(B \cup Y) = \frac{2}{3}$ violating null-additivity.

We conclude this appendix with the proof of Proposition 3.2, which finds the support of a neo-additive capacity.

Proof of Proposition 3.2: Clearly $\alpha\mu(\varnothing) + (1-\alpha)\bar{\mu}(\varnothing) = 0 = \nu(\varnothing)$ and $\alpha\mu(S_{-i}) + (1-\alpha)\bar{\mu}(S_{-i}) = 1 = \nu(S_{-i})$. If $\varnothing \subsetneqq A \subsetneqq S_{-i}$ then $\alpha\mu(A) + (1-\alpha)\bar{\mu}(A) = \alpha(1-\delta)\pi(A) + (1-\alpha)(1-\delta)\pi(A) + (1-\alpha)\delta \cdot 1 = \delta(1-\alpha) + (1-\delta)\pi(A) = \nu(A)$.
If $\varnothing \subsetneqq A \subsetneqq S_{-i}$ then $\bar{\mu}(\neg A) - \mu(A) = [1-(1-\delta)\pi(\neg A)] - (1-\delta)\pi(A) = 1-(1-\delta)\pi(A) - (1-\delta)\pi(\neg A) = \delta$ since $\pi(A) + \pi(\neg A) = 1$.

By definition $\operatorname{supp}\nu_{JP} = \operatorname{supp}\mu$. Suppose that $\hat{s} \in \operatorname{supp}\pi$. If $p \in \mathcal{C}(\mu)$ then $p(\hat{s}) \geqslant \mu(\hat{s}) = (1-\delta)\pi(\hat{s}) > 0$. Thus $\forall p \in \mathcal{C}(\mu), p(\hat{s}) > 0$, which implies $\hat{s} \in \operatorname{supp}\mu$, hence $\operatorname{supp}\pi \subseteq \operatorname{supp}\mu$.

To show $\operatorname{supp}\mu \subseteq \operatorname{supp}\pi$, suppose if possible, there exists $\tilde{s} \in \operatorname{supp}\mu \backslash \operatorname{supp}\pi$. Then $\mu(\tilde{s}) = (1-\delta)\pi(\tilde{s}) = 0$. If $q = \operatorname{argmin}_{p\in\mathcal{C}(\mu)} p(\tilde{s})$, then $q(\tilde{s}) = 0$. Thus $\tilde{s} \notin \operatorname{supp} q$ and consequentially $\tilde{s} \notin \operatorname{supp}\mu = \bigcap_{p\in\mathcal{C}(\mu)} \operatorname{supp} p$. However this is a contradiction, which establishes that $\operatorname{supp}\mu \subseteq \operatorname{supp}\pi$. The result follows. ■

B EXAMPLES

Lemma B.1 *In the game of Example 3.1 the neo-additive capacities* $(\widehat{\nu}_1, \widehat{\nu}_2, \widehat{\nu}_3)$ *with* $\widehat{\pi}_1((R,X)) = 1$, $\widehat{\pi}_2((U,Y)) = 1$, $\widehat{\pi}_3((U,R)) = 1$, *are an EUA if* $3(1-\delta_1) - 5\alpha_1\delta_1 \geq 0$, $3(1-\delta_2) - 5\alpha_2\delta_2 \geq 0$, *holds. If both inequalities are reversed, then there are two pure EUA* $(\widetilde{\nu}_1, \widetilde{\nu}_2, \widetilde{\nu}_3)$

[32] Condition (ii)(d) in Proposition 3.1 in Chateauneuf, Eichberger, and Grant [6].

with $\widetilde{\pi}_1((L,X)) = 1, \quad \widetilde{\pi}_2((D,X)) = 1, \quad \widetilde{\pi}_3((D,L)) = 1,$ *and* $(\bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3)$ *with* $\bar{\pi}_1((L,Y)) = 1, \quad \bar{\pi}_2((D,Y)) = 1, \quad \bar{\pi}_3((D,L)) = 1.$

Proof. Both types of behaviour can be EUA depending on the ambiguity-attitude of the players and their degrees of ambiguity regarding the behaviour of their opponents.

Representing beliefs by neo-additive capacities, it is not difficult to derive the CEU of the players:

$$\begin{aligned}
V_1(U) = &\quad 3\left[(1-\alpha_1)\delta_1 + (1-\delta_1)\pi_1(RX, RY)\right], \\
V_1(D) = &\quad 8(1-\alpha_1)\delta_1 + (1-\delta_1)[8\pi_1(RY) + \pi_1(LX, LY)], \\
\\
V_2(L) = &\quad 8(1-\alpha_2)\delta_2 + (1-\delta_2)\left[8\pi_2(UX) + \pi_2(DX, DY)\right], \\
V_2(R) = &\quad 3\left[(1-\alpha_2)\delta_2 + (1-\delta_2)\pi_2(UX, UY)\right], \\
\\
V_3(X) = &\quad 3(1-\alpha_3)\delta_3 + (1-\delta_3)\left[3\pi_3(UR) + \pi_3(DL)\right], \\
V_3(Y) = &\quad 3(1-\alpha_3)\delta_3 + (1-\delta_3)\left[3\pi_3(UR) + \pi_3(DL)\right].
\end{aligned}$$

If $(\alpha_1, \alpha_2, \alpha_3)$ and $(\delta_1, \delta_2, \delta_3)$ satisfy $3(1-\delta_1) - 5\alpha_1\delta_1 \geq 0$ and $3(1-\delta_2) - 5\alpha_2\delta_2 \geq 0$, the neo-additive capacities $(\widehat{\nu}_1, \widehat{\nu}_2, \widehat{\nu}_3)$ with $\widehat{\pi}_1((R,X)) = 1,\ \widehat{\pi}_2((U,Y)) = 1,\ \widehat{\pi}_3((U,R)) = 1$, are an EUA, i.e., $\varnothing \neq \operatorname{supp} \hat{\nu}_i \subseteq \underset{j \neq i}{\times} R_j(\hat{\nu}_j)$

$$\begin{aligned}
&V_1(U) - V_1(D) = 3(1-\delta_1) - 5(1-\alpha_1)\delta_1 && \text{for} \quad \pi_1((R,X)) = 1, \\
&V_2(R) - V_2(L) = 3(1-\delta_2) - 5(1-\alpha_2)\delta_2 && \text{for} \quad \pi_2((U,Y)) = 1, \\
&V_3(X) - V_3(Y) = 0 && \text{for} \quad \pi_3((U,R)) = 1.
\end{aligned}$$

The following beliefs are an EUA

- Equilibrium beliefs of Player 1: $\mu_1(E) = \begin{cases} 1-\delta_1 & \text{for} \quad \{(R,X)\} \subseteq E \subsetneqq S_{-i}, \\ 0 & E \cap (R,X) = \varnothing, \end{cases}$
- Equilibrium beliefs of Player 2: $\mu_2(E) = \begin{cases} 1-\delta_2 & \text{for} \quad \{(U,Y)\} \subseteq E \subsetneqq S_{-i}, \\ 0 & E \cap (U,Y) = \varnothing, \end{cases}$
- Equilibrium beliefs of Player 3: $\mu_3(E) = \begin{cases} 1-\delta_3 & \text{for} \quad \{(U,R)\} \subseteq E \subsetneqq S_{-i}, \\ 0 & E \cap (U,R) = \varnothing. \end{cases}$

For these beliefs, one obtains the following best responses:

- Best reply of Player 1: $R_1(\nu_1) = \begin{cases} \{U\} & \text{for} \quad 3(1-\delta_1) - 5\alpha_1\delta_1 \geq 0, \\ \{D\} & \text{otherwise,} \end{cases}$
- Best reply of Player 2: $R_2(\nu_2) = \begin{cases} \{R\} & \text{for} \quad 3(1-\delta_2) - 5\alpha_2\delta_2 \geq 0, \\ \{L\} & \text{otherwise,} \end{cases}$
- Best reply of Player 3: $R_3(\nu_3) = \{X, Y\} \quad \text{for all} \quad \nu_3.$

If the conditions in equation (2) both fail, then there are two pure EUA $(\widetilde{\nu}_1, \widetilde{\nu}_2, \widetilde{\nu}_3)$ with $\widetilde{\pi}_1((L,X)) = 1$, $\widetilde{\pi}_2((D,X)) = 1$, $\widetilde{\pi}_3((D,L)) = 1$, and $(\bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3)$ with $\bar{\pi}_1((L,Y)) = 1$, $\bar{\pi}_2((D,Y)) = 1$, $\bar{\pi}_3((D,L)) = 1$, which correspond to the standard Nash equilibria of this game. ■

C GAMES WITH AMBIGUITY

This appendix contains proofs of our existence and comparative statics results and some supplementary results. It uses techniques from Topkis [54].

C.1 Existence

We start with a preliminary definition and Lemma.

Definition C.1 *Suppose that B is a correspondence from a partially ordered set S to a lattice T such that for all $s \in S, B(s)$ is a sub-lattice of T, then we say that B is* increasing *if when $\hat{s} > \tilde{s}$, and $\hat{t} \in B(\hat{s}), \tilde{t} \in B(\tilde{s})$ then $\min\{\hat{t}, \tilde{t}\} \in B(\tilde{s})$ and $\max\{\hat{t}, \tilde{t}\} \in B(\hat{s})$.*

Lemma C.1 *Let S be a lattice and let $\beta : S \to S$ be an increasing correspondence. Then*

1. *β has a fixed point;*
2. *$\sup\{s : \beta(s) \geqslant s\}$ is the greatest fixed point of β.*

Proof. Let $\mathcal{T} = \{s : \beta(s) \geqslant s\}$. Note that $\mathcal{T}$ is non-empty since $\underline{s} \in \mathcal{T}$, (where $\underline{s} = \min S$). Let $s' = \sup \mathcal{T}$. By definition if $s'' > s'$ then

$$\beta(s'') < s''. \tag{3}$$

Suppose $\tilde{s} \in \mathcal{T}$, then $\beta(\tilde{s}) \geqslant \tilde{s}$. Since β is increasing, $\beta(s') \geqslant \beta(\tilde{s})$ and $\beta(s') \geqslant s' \geqslant \tilde{s}$. Thus $\beta(\beta(s')) \geqslant \beta(s') \geqslant s'$, which implies $\beta(s') \in \mathcal{T}$ and hence $s' \geqslant \beta(s') \geqslant s'$.[33] The last equation implies that s' is a fixed point of β. Equation (3) implies that there is no greater fixed point. ■

Proof of Theorem 3.1 Choose an n-tuple of parameters $\delta = \langle \delta_1, ..., \delta_n \rangle$ such that $\lambda \geqslant \delta \geqslant \gamma$. Let $\nu_i^{\hat{s}_{-i}}$ denote the neo-additive capacity on S_{-i} defined by $\nu_i^{\hat{s}_{-i}}(S_{-i}) = 1, \nu_i^{\hat{s}_{-i}}(A) = \delta_i \alpha_i$, if $\hat{s}_{-i} \notin A, \nu_i^{\hat{s}_{-i}}(A) = \delta_i \alpha + 1 - \delta_i$, otherwise. (Informally $\nu_i^{\hat{s}_{-i}}$ represents a situation where i has an ambiguous belief that his/her opponents will play $\hat{s}_{-i}$.) Define $V_i(s_i, s_{-i})$ to be player i's (Choquet) expected utility from playing s_i when his/her beliefs are represented by $\nu_i^{\hat{s}_{-i}}$ i.e.

$$V_i(s_i, s_{-i}) = \int u_i(s_i, s_{-i})\, d\nu_i^{\hat{s}_{-i}}(s_{-i}) = \delta_i \alpha_i u_i(s_i, \bar{s}_{-i}) + \delta_i (1 - \alpha_i) u_i(s_i, \underline{s}_{-i}) + (1 - \delta_i) u_i(s_i, s_{-i}).$$

Lemma C.6 implies that $V_i(s_i, s_{-i})$ has increasing differences in $\langle s_i, s_{-i} \rangle$.

Define $\beta_i(\hat{s}_{-i}) = \operatorname{argmax}_{s_i \in S_i} V_i(s_i, s_{-i})$ and $\beta(s) = \langle \beta_1(s_{-1}), ..., \beta_n(s_{-n}) \rangle$. Thus $\beta_i(\hat{s}_{-i})$ is the best response of player i, if his/her beliefs are a neo-additive capacity which represents an ambiguous belief that his/her opponents will play $\hat{s}_{-i}$. Since $V_i(s_i, s_{-i})$ has increasing differences in $\langle s_i, s_{-i} \rangle$, β_i is an increasing correspondence. (The proof is similar to that of Lemma C.8.) Thus by Lemma C.1, β has a fixed point $\hat{s}$. This implies $\nu^{\hat{s}} = \left\langle \nu_1^{\hat{s}_{-1}}, ..., \nu_n^{\hat{s}_{-n}} \right\rangle$ is a pure equilibrium. By Proposition 3.2, $\nu_i^{\hat{s}_{-i}}$ may be written in the form $\nu_i^{\hat{s}_{-i}} = \alpha_i \mu_i + (1 - \alpha_i) \bar{\mu}_i$, where μ_i is convex and $\lambda(\mu_i) = \gamma(\mu_i) = \delta_i$. ■

C.2 Comparative Statics Proofs

C.2.1 Correspondences on Partially Ordered Sets

This section contains some results about increasing correspondences and selections from them.

Lemma C.2 *Suppose that B_λ is an increasing correspondence from a partially ordered set S to a totally ordered set T for all λ in an index set Λ, then $\bar{B}(s) = \max_{\lambda \in \Lambda} B_\lambda(s)$ and $\underline{B}(s) = \min_{\lambda \in \Lambda} B_\lambda(s)$ are increasing functions from S to T.*

Proof. Suppose that $\hat{s} > \tilde{s}$. Then there exists $\tilde{\lambda} \in \Lambda$ such that $\bar{B}(\tilde{s}) = B_{\tilde{\lambda}}(\tilde{s})$. Since $B_{\tilde{\lambda}}$ is increasing, $\bar{B}(\hat{s}) \succcurlyeq B_{\tilde{\lambda}}(\hat{s}) \succcurlyeq B_{\tilde{\lambda}}(\tilde{s}) = \bar{B}(\tilde{s})$, which demonstrates that $\bar{B}$ is increasing.

[33] $\beta(\beta(s')) = \{s : \exists \tilde{s} \in \beta(s'), s = \beta(\tilde{s})\}$.

There exists $\hat{\lambda} \in \Lambda$ such that $\underline{B}(\hat{s}) = B_{\hat{\lambda}}(\hat{s}) = \min B_{\hat{\lambda}}(\hat{s})$. Since $B_{\hat{\lambda}}$ is increasing, $B_{\hat{\lambda}}(\hat{s}) \succcurlyeq B_{\hat{\lambda}}(\tilde{s})$. Finally $B_{\hat{\lambda}}(\tilde{s}) \succcurlyeq \underline{B}(\tilde{s})$, which establishes that $\underline{B}(\hat{s}) \succcurlyeq \underline{B}(\tilde{s})$. ■

The following lemma describes some properties of fixed points of functions on partially ordered sets.

Lemma C.3 *Let S and A be partially ordered sets and let $f : S \times A \to S$ be a function which is increasing in s and α. Then the greatest fixed point of $f(\cdot, \alpha)$ is an increasing function of α.*

Proof. Let $s(\alpha)$ denote the greatest fixed point of $f(\cdot, \alpha)$. Since f is increasing in α, if $\hat{\alpha} > \tilde{\alpha}$, $\{s : f(s, \tilde{\alpha}) \geqslant s\} \subseteq \{s : f(s, \hat{\alpha}) \geqslant s\}$. Hence $s(\hat{\alpha}) = \sup\{s : f(s, \hat{\alpha}) \geqslant s\} \geqslant \sup\{s : f(s, \tilde{\alpha}) \geqslant s\} = s(\tilde{\alpha})$ by Lemma C.1. ■

C.2.2 Constant Contamination Capacities

Below we define a special case of JP capacities which arise naturally when considering pure equilibria in games.

Definition C.2 (Constant Contamination CC) *A capacity, $\nu_i^{\tilde{s}}(\alpha_i, \delta_i, \varsigma_i)$, on S_{-i} is said to display* constant contamination *if it may be written in the form $\nu_i^{\tilde{s}}(\alpha_i, \delta_i, \varsigma_i) = \nu_i^{\tilde{s}_{-i}}(A, \alpha_i, \delta_i, \varsigma_i) = (1-\delta_i)\pi_i^{\tilde{s}}(A) + \delta_i[\alpha_i\varsigma_i(A) + (1-\alpha_i)\bar{\varsigma}_i(A)]$, where $\pi_i^{\tilde{s}}$ denotes the probability distribution*[34] *on S_{-i}, which assigns probability 1 to $\tilde{s}_{-i}$ and ς_i is a convex capacity with* $\operatorname{supp}\varsigma_i = \emptyset$. *To simplify notation we shall suppress the arguments $(\alpha_i, \delta_i, \varsigma_i)$ when it is convenient.*

We interpret the capacity $\nu_i(\varsigma_i, \delta_i, \alpha_i)$ as describing a situation where player i 'believes' that his/her opponents will play the pure strategy profile $\tilde{s}_{-i}$ but lacks confidence in this belief. The CC-capacity has a separation between beliefs represented by π_i, ambiguity-attitude represented by α_i and ambiguity represented by ς_i and δ_i. The parameter δ_i determines the weight the individual gives to ambiguity. Higher values of δ_i correspond to more ambiguity. The capacity ς_i determines which strategy profiles the player regards as ambiguous. The following result finds the support of a CC capacity.

Lemma C.4 *Let $\nu_i = (1-\delta_i)\pi^{\tilde{s}}(A) + \delta_i[\alpha_i\varsigma_i(A) + (1-\alpha_i)\bar{\varsigma}_i(A)]$ be a CC capacity. Then* $\operatorname{supp}_{JP}\nu_i = \{\tilde{s}_{-i}\}$.

[34] This distribution is usually denoted by $\delta_{\tilde{s}}$ However we are using the symbol δ elsewhere to denote degree of ambiguity.

Proof. If we define a convex capacity $\mu_i(A)$ by $\mu_i = (1-\delta_i)\pi^{\tilde{s}}(A) + \delta_i\varsigma_i(A)$ then $\nu_i = \alpha_i\mu_i + (1-\alpha_i)\bar{\mu}_i$. By definition $\operatorname{supp}_{JP}\nu_i = \operatorname{supp}\mu_i$. If $p \in \mathcal{C}(\mu_i)$ then $p(\tilde{s}) \geqslant \mu_i(\tilde{s}) = (1-\delta_i)\pi^{\tilde{s}}(\tilde{s}) + \delta_i\varsigma_i(\tilde{s}) = (1-\delta_i)$, since $\operatorname{supp}\varsigma_i = \varnothing$. Thus $\forall p \in \mathcal{C}(\mu_i), p(\tilde{s}) > 0$, which implies $\tilde{s} \in \operatorname{supp}\mu$.

To show $\{\tilde{s}\} = \operatorname{supp}\mu$, suppose if possible, there exists $\bar{s} \in \operatorname{supp}\mu$ such that $\tilde{s} \neq \bar{s}$. Then $\mu_i(\bar{s}) = (1-\delta_i)\pi^{\tilde{s}}(\bar{s}) + \delta_i\varsigma_i(\bar{s}) = 0$ since $\pi^{\tilde{s}}(\bar{s}) = 0$ and $\operatorname{supp}\varsigma_i = \varnothing$. If $q = \operatorname{argmin}_{p\in\mathcal{C}(\mu)} p(\bar{s})$, then $q(\bar{s}) = 0$. Thus $\bar{s} \notin \operatorname{supp} q$ and consequentially $\bar{s} \notin \operatorname{supp}\mu_i = \bigcap_{p\in\mathcal{C}(\mu_i)} \operatorname{supp} p$. However this is a contradiction, which establishes that $\operatorname{supp}\mu_i \subseteq \{\tilde{s}\}$. The result follows. ■

The following lemma shows that any equilibrium capacity has the constant contamination form.

Lemma C.5 *Let Γ be a game with positive externalities and let $\hat{\nu}$ be a pure equilibrium in JP capacities of Γ with equilibrium strategy profile $\hat{s}$. Then $\hat{\nu}$ is a profile of CC-capacities, i.e. there exist convex capacities $\zeta_i, 1 \leqslant i \leqslant n$, with $\operatorname{supp}\zeta_i = \varnothing$ and $\delta_i, 1 \leqslant i \leqslant n$, such that if we define $\mu_i = \delta_i\varsigma_i + (1-\delta_i)\pi_i^{\hat{s}_{-i}}$ then $\nu_i = \alpha\mu_i + (1-\alpha_i)\bar{\mu}_i$ for $1 \leqslant i \leqslant n$. Moreover $\lambda(\mu) = (1-\delta_i)\lambda(\zeta)$ and $\gamma(\mu) = (1-\delta_i)\gamma(\zeta)$.*

Proof. Since $\hat{\nu}$ is an equilibrium in JP-capacities, we may write the equilibrium beliefs of individual i in the form $\hat{\nu}_i = \alpha_i\mu_i + (1-\alpha_i)\bar{\mu}_i$ for some convex capacity μ_i. Define a capacity ζ_i by $\zeta_i = \frac{\mu_i - \delta_i\pi_i^{\hat{s}_{-i}}}{1-\delta_i}$, where $\delta_i = \mu_i(\hat{s}_{-i})$. Then $\hat{\nu}_i = \delta_i\pi_i^{\hat{s}_{-i}}(A) + (1-\delta_i)[\alpha_i\varsigma_i(A) + (1-\alpha_i)\bar{\varsigma}_i(A)]$. We claim that ζ_i is convex. To show this we need to show $\zeta_i(A\cup B) \geqslant \zeta_i(A) + \zeta_i(B) - \zeta_i(A\cap B)$ for all $A, B \subseteq S_{-i}$. There are four cases to consider.

If $\hat{s}_{-i} \in A$ and $\hat{s}_{-i} \in B$, then $\zeta_i(A\cup B) + \zeta_i(A\cap B) = \frac{1}{1-\delta_i}(\mu_i(A\cup B) + \mu_i(A\cap B) - 2\delta_i) \geqslant \frac{1}{1-\delta_i}(\mu_i(A) + \mu_i(B) - 2\delta_i)$ by convexity of μ_i. Since $\zeta_i(A) + \zeta_i(B) = \frac{1}{1-\delta_i}(\mu_i(A) + \mu_i(B) - 2\delta_i)$ the claim is proved in this case.

If $\hat{s}_{-i} \notin A$ and $\hat{s}_{-i} \notin B$, then the claim follows from convexity of μ_i, since $\zeta_i = \frac{1}{1-\delta_i}\mu_i$ for all four sets.

If $\hat{s}_{-i} \in A$ and $\hat{s}_{-i} \notin B$, $\zeta_i(A\cup B) + \zeta_i(A\cap B) = \frac{1}{1-\delta_i}(\mu_i(A\cup B) - \delta_i) + \frac{1}{1-\delta_i}\mu_i(A\cap B) \geqslant \frac{1}{1-\delta_i}\mu_i(A) + \frac{1}{1-\delta_i}\mu_i(B) - \frac{\delta_i}{1-\delta_i}$, by convexity of μ_i. Since $\zeta_i(A) + \zeta_i(B) = \frac{1}{1-\delta_i}(\mu_i(A) + \mu_i(B) - \delta_i)$ this proves convexity in this case. The remaining case is similar.

Since $\operatorname{supp}\nu_i = \hat{s}_{-i}$, $\operatorname{supp}\mu_i = \hat{s}_{-i}$. Hence for $\tilde{s}_{-i} \neq \hat{s}_{-i}$, $\mu_i(\tilde{s}_{-i}) = 0$, which implies that for all $s_{-i} \in S_{-i}, \zeta_i(s_{-i}) = 0$. Since ζ_i is convex it follows from Proposition A.1 that $\operatorname{supp}\zeta_i = \varnothing$.

Now consider $A \subsetneqq S_{-i}$. Either $\hat{s}_{-i} \in A$ or $\hat{s}_{-i} \in \neg A$. Without loss of generality $\hat{s}_{-i} \in A$. Then $1 - \zeta_i(A) - \zeta_i(\neg A) = 1 - \frac{1}{1-\delta_i}(\mu_i(A) - \delta_i) - \frac{1}{1-\delta_i}(\mu_i(\neg A))$

$= \frac{1}{1-\delta_i}[1 - \delta_i - \mu_i(A) + \delta_i - \mu_i(\neg A)] = \frac{1}{1-\delta_i}[1 - \mu_i(A) - \mu_i(\neg A)]$, which implies $(1-\delta_i)\lambda(\zeta) = \lambda(\mu)$ and $(1-\delta_i)\gamma(\zeta) = \gamma(\mu)$. ■

C.2.3 Increasing/decreasing Differences

Recall that a game, $\Gamma = \langle N; (S_i), (u_i) : 1 \leqslant i \leqslant n \rangle$, has *positive aggregate externalities* if $u_i(s_i, s_{-i}) = u_i(s_i, f_i(s_{-i}))$, for $1 \leqslant i \leqslant n$, where u_i is increasing (resp. decreasing) in f_i and $f_i : S_{-i} \to \mathbb{R}$ is increasing in all arguments. Since S_{-i} is finite, we may enumerate the possible values of f_i, $f_i^0 < ... < f_i^M$. Since f is increasing $f_i^0 = f(\underline{s}_1, ..., \underline{s}_n)$ and $f_i^M = f(\bar{s}_1, ..., \bar{s}_n)$. The *Choquet integral* of $u_i(s_i, s_{-i})$ with respect to capacity ν_i on S_{-i} may be written in the form:

$$V_i(s_i) = \int u_i(s_i, s_{-i})\, d\nu_i = u_i(s_i, f_M)\,\nu_i(H_M) + \sum_{r=0}^{M-1} u_i(s_i, f_r)\,[\nu_i(H_r) - \nu_i(H_{r+1})],$$

where H_r denotes the event $\{s_{-i} \in S_{-i} : f(s_{-i}) \geqslant f_r\}$.

Define $W_i(s_i, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i)$ to be player i's (Choquet) expected payoff given that his/her beliefs are represented by the capacity $\nu_i^{\tilde{s}_{-i}}(A, \alpha_i, \delta_i, \varsigma_i)$ i.e.

$$W_i(s_i, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i) = \int u_i(s_i, s_{-i})\, d\nu_i^{\tilde{s}_{-i}}(A, \alpha_i, \delta_i, \varsigma_i)$$
$$= (1-\delta_i)\, u_i(s_i, \tilde{s}_{-i}) + \alpha_i\delta_i \int u_i(s_i, s_{-i})\, d\varsigma_i + (1-\alpha_i)\,\delta_i \int u_i(s_i, s_{-i})\, d\bar{\varsigma}_i.$$

Lemma C.6 *If $u_i(s_i, s_{-i})$ satisfies increasing differences in $\langle s_i, s_{-i} \rangle$ so does $W_i(s_i, s_{-i}, \alpha_i, \delta_i, \varsigma_i)$.*

Proof. Suppose $s'_{-i} > s''_{-i}$, then $W_i(s_i, s'_{-i}, \alpha_i, \delta_i, \varsigma_i) - W_i(s_i, s''_{-i}, \alpha_i, \delta_i, \varsigma_i)$

$= \alpha_i\delta_i \int u_i(s_i, s_{-i})\, d\varsigma_i + (1-\alpha_i)\,\delta_i \int u_i(s_i, s_{-i})\, d\bar{\varsigma}_i + (1-\delta_i)\, u_i(s_i, s'_{-i}) - \alpha_i\delta_i \int u_i(s_i, s_{-i})\, d\varsigma_i$
$- (1-\alpha_i)\,\delta_i \int u_i(s_i, s_{-i})\, d\bar{\varsigma}_i - (1-\delta_i)\, u_i(s_i, s''_{-i}) = (1-\delta_i)\left[u_i(s_i, s'_{-i}) - u_i(s_i, s''_{-i})\right]$, which is increasing in s_i since u_i has increasing differences in s_i, s_{-i}. ■

Lemma C.7 *The function $W_i(s_i, s_{-i}, \alpha_i, \delta_i, \varsigma_i)$ satisfies decreasing differences in $\langle s_i, \alpha_i \rangle$.*

Proof. Suppose $s'_i > s''_i$, then $W_i(s'_i, s_{-i}, \alpha_i, \delta_i, \varsigma_i) - W_i(s''_i, s_{-i}, \alpha_i, \delta_i, \varsigma_i)$

$= \alpha_i\delta_i \int u_i(s'_i, s_{-i})\, d\varsigma_i + (1-\alpha_i)\,\delta_i \int u_i(s'_i, s_{-i})\, d\bar{\varsigma}_i + (1-\delta_i)\, u_i(s'_i, s_{-i})$
$-\alpha_i\delta_i \int u_i(s''_i, s_{-i})\, d\varsigma_i - (1-\alpha_i)\,\delta_i \int u_i(s''_i, s_{-i})\, d\bar{\varsigma}_i - (1-\delta_i)\, u_i(s''_i, s_{-i})$

$$= \alpha_i \delta_i \left[\int \left[u_i \left(s_i', s_{-i} \right) - u_i \left(s_i'', s_{-i} \right) \right] d\varsigma_i - \int \left[u_i \left(s_i', s_{-i} \right) - u_i \left(s_i'', s_{-i} \right) \right] d\bar{\varsigma}_i \right]$$
$$+ \delta_i \int \left[u_i \left(s_i', s_{-i} \right) - u_i \left(s_i'', s_{-i} \right) \right] d\bar{\varsigma}_i + \left(1 - \delta_i \right) \left[u_i \left(s_i', s_{-i} \right) - u_i \left(s_i'', s_{-i} \right) \right].$$

We have used the fact that since there are positive aggregate externalities, all four integrands in the square brackets are comonotonic. It is sufficient to show that the coefficient of α_i is positive. This is equal to

$$\begin{aligned} & \left[u_i \left(s_i', f_M \right) - u_i \left(s_i'', f_M \right) \right] \varsigma_i \left(s_{-i}^M \right) + \sum_{r=0}^{M-1} \left[u_i \left(s_i', s_{-i}^r \right) - u_i \left(s_i'', s_{-i}^r \right) \right] \left[\varsigma_i \left(H_r \right) - \varsigma_i \left(H_{r+1} \right) \right] \\ & - \left[u_i \left(s_i', f_M \right) - u_i \left(s_i'', f_M \right) \right] \bar{\varsigma}_i \left(s_{-i}^M \right) \\ & - \sum_{r=0}^{M-1} \left[u_i \left(s_i', s_{-i}^r \right) - u_i \left(s_i'', s_{-i}^r \right) \right] \left[\bar{\varsigma}_i \left(H_r \right) - \bar{\varsigma}_i \left(H_{r+1} \right) \right]. \end{aligned} \tag{4}$$

By increasing differences $u_i \left(s_i', s_{-i} \right) - u_i \left(s_i'', s_{-i} \right) > 0$ and is an increasing function of s_i. Equation (4) is the difference of two weighted sums of $u_i \left(s_i', s_{-i} \right) - u_i \left(s_i'', s_{-i} \right)$. The weights in the first sum are first order stochastically dominated by those in the second, (proved below). Thus the first sum is smaller which makes the overall expression negative. This establishes that $W_i \left(s_i', s_{-i}, \alpha_i \right) - W_i \left(s_i'', s_{-i}, \alpha_i \right)$ is a decreasing function of α_i.

Proof of stochastic dominance The first k weights in the first sum add up to: $\varsigma_i \left(s_{-i}^M \right) + \sum_{r=M-k+1}^{M-1} \left[\varsigma_i \left(H_r \right) - \varsigma_i \left(H_{r+1} \right) \right] = \varsigma_i \left(H_{M-k+1} \right)$. Similarly the first k weights in the second sum in total are equal to: $\bar{\varsigma}_i \left(H_{M-k+1} \right) = 1 - \varsigma_i \left(\neg H_{M-k+1} \right)$. Since ς_i is convex, $\varsigma_i \left(H_{M-k+1} \right) \leqslant 1 - \varsigma_i \left(\neg H_{M-k+1} \right)$, which establishes first order stochastic dominance. ■

Lemma C.8 *The the best response correspondence of player i, $B_i \left(\hat{s}_{-i}, \alpha_i, \delta_i, \varsigma_i \right)$, defined by $B_i \left(s_{-i}, \alpha_i, \delta_i, \varsigma_i \right) = \operatorname{argmax}_{s_i \in S_i} W_i \left(s_i, s_{-i}, \alpha_i, \delta_i, \varsigma_i \right)$ is increasing in s_{-i} and decreasing in α_i.*

Proof. To show $B_i \left(s_{-i}, \alpha_i, \delta_i, \varsigma_i \right)$ is increasing in s_{-i}, assume $\hat{s}_{-i} > \tilde{s}_{-i}$. Consider $y \in B_i \left(\tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i \right), z \in B_i \left(\hat{s}_{-i}, \alpha_i, \delta_i, \varsigma_i \right)$ and let $m = \min \{ y, z \}$ and $M = \max \{ y, z \}$. Now $W_i \left(y, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i \right) \geqslant W_i \left(z, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i \right)$ which implies $W_i \left(M, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i \right) \geqslant W_i \left(z, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i \right)$. By increasing differences, $W_i \left(M, \hat{s}_{-i}, \alpha_i, \delta_i, \varsigma_i \right) \geqslant W_i \left(z, \hat{s}_{-i}, \alpha_i, \delta_i, \varsigma_i \right)$, hence $M \in B_i \left(\hat{s}_{-i}, \alpha_i, \varsigma_i \right)$.

Since $W_i \left(y, \hat{s}_{-i}, \alpha_i, \delta_i, \varsigma_i \right) - W_i \left(z, \hat{s}_{-i}, \alpha_i, \delta_i, \varsigma_i \right) \leqslant 0$, increasing differences implies $W_i \left(y, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i \right) - W_i \left(z, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i \right) \leqslant 0$. Thus $W_i \left(m, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i \right) \geqslant W_i \left(y, \tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i \right)$ and hence $m \in B_i \left(\tilde{s}_{-i}, \alpha_i, \delta_i, \varsigma_i \right)$. This establishes that $B_i \left(s_{-i}, \alpha_i, \delta_i, \varsigma_i \right)$ is increasing in s_{-i}. We may establish that $B_i \left(s_{-i}, \alpha_i, \varsigma_i \right)$ is decreasing in α_i by a similar argument. ■

Definition C.3 *The maximal and minimal best response correspondences of player i are defined respectively by*

$$\begin{aligned}
\bar{B}_i\left(s_{-i},\alpha_i,\gamma_i,\lambda_i,\delta_i\right) &= \max_{\zeta_i}\left\{B_i\left(s_{-i},\alpha_i,\delta_i,\zeta_i\right);\forall A\subsetneqq S_{-i},\frac{\lambda_i}{1-\delta_i}\geqslant\bar{\zeta}_i\left(\neg A\right)-\zeta_i\left(A\right)\geqslant\frac{\gamma_i}{1-\delta_i}\right\},\\
\underline{B}_i\left(s_{-i},\alpha_i,\gamma_i,\lambda_i,\delta_i\right) &= \min_{\zeta_i}\left\{B_i\left(s_{-i},\alpha_i,\delta_i,\zeta_i\right);\forall A\subsetneqq S_{-i},\frac{\lambda_i}{1-\delta_i}\geqslant\bar{\zeta}_i\left(\neg A\right)-\zeta_i\left(A\right)\geqslant\frac{\gamma_i}{1-\delta_i}\right\}.
\end{aligned}$$

It follows from Lemma C.5 that the maximal best response correspondence is the greatest best response to all beliefs whose support is the pure strategy s_{-i} with minimal (resp. maximal) degree of ambiguity is at least γ (resp. at most λ).

Proof of Theorem 4.1 We shall only prove the result for the highest equilibrium strategy. The lowest equilibrium strategy can be covered by a similar argument. Lemma C.5 establishes that if $\hat{s}$ is an equilibrium strategy profile when the minimal (resp. maximal) degree of ambiguity is γ (resp. λ), then there exist ζ_i with $\frac{\lambda_i}{1-\delta_i}\geqslant\bar{\zeta}_i\left(\neg A\right)-\zeta_i\left(A\right)\geqslant\frac{\gamma_i}{1-\delta_i}$ such that $\hat{s}_i\in B_i\left(s_{-i},\alpha_i,\zeta_i\right)$ for $1\leqslant i\leqslant n$. Thus any given equilibrium satisfying these constraints is smaller than the largest fixed point of the maximal best response correspondence $\bar{B}_i\left(s_{-i},\alpha_i,\gamma_i,\lambda_i\right)$.

Therefore since $\bar{s}\left(\alpha\right)$ is the profile of greatest equilibrium strategies it is the largest fixed point of the maximal best response function, i.e. $\bar{s}\left(\alpha\right)\in\bar{B}\left(\bar{s},\bar{\alpha}\right)$ and $\underline{s}\left(\alpha\right)\in\underline{B}\left(\underline{s},\underline{\alpha}\right)$. By Lemma C.8, $\bar{B}_i\left(s_{-i},\alpha_i,\gamma_i,\lambda_i\right)$ is increasing in s_{-i} and decreasing α_i. It follows from Lemma C.3 that $\bar{s}\left(\alpha\right)$ is decreasing in α. ■

C.2.4 Multiple Equilibria

In this section we show that equilibrium is unique if there is sufficient ambiguity.

Lemma C.9 *Consider a game, Γ, of positive externalities and increasing differences. There exists $\bar{\gamma}$ such that if the minimal degree of ambiguity is $\gamma\left(\mu_i\right)\geqslant\bar{\gamma}$, then in any equilibrium $\nu=\left\langle\nu_1,...,\nu_n\right\rangle$, $\operatorname{supp}\nu_i\subseteq\mathcal{A}$, where $\mathcal{A}$ denotes the set* $\operatorname{argmax}_{s_i\in S_i}\left\{\alpha_i u_i\left(s_i,\underline{s}_{-i}\right)+\left(1-\alpha_i\right)u_i\left(s_i,\bar{s}_{-i}\right)\right\}$, *for $1\leqslant i\leqslant n$.*

Proof. Suppose $\hat{s}_i\in\mathcal{A},\tilde{s}_i\notin\mathcal{A}$. Number the strategy profiles of the opponents so that $u_i\left(\hat{s}_i,s^1_{-i}\right)\geqslant u_i\left(\hat{s}_i,s^2_{-i}\right)\geqslant...\geqslant u_i\left(\hat{s}_i,s^R_{-i}\right)$ and $u_i\left(\tilde{s}_i,\sigma^1_{-i}\right)\geqslant u_i\left(\tilde{s}_i,\sigma^2_{-i}\right)\geqslant...\geqslant u_i\left(\tilde{s}_i,\sigma^R_{-i}\right)$. Although in general $\sigma^r_{-i}\neq s^r_{-i}$, positive externalities implies that $s^1_{-i}=\sigma^1_{-i}=\bar{s}_{-i}$ and $s^R_{-i}=$

$\sigma_{-i}^{R} = \underline{s}_{-i}$. Suppose that the beliefs of individual i may be represented by a JP-capacity, $\nu_i = \alpha_i \mu_i + (1-\alpha_i)\bar{\mu}_i$. If i plays strategy $\hat{s}_i$, (s)he receives utility:

$$V_i(\hat{s}_i) = \alpha_i \int u_i(\hat{s}_i, s_{-i})\, d\mu_i + (1-\alpha_i) \int u_i(\hat{s}_i, s_{-i})\, d\bar{\mu}_i = \alpha_i u_i(\hat{s}_i, \bar{s}_{-i})\, \mu_i(\bar{s}_{-i})$$
$$+\alpha_i \textstyle\sum_{r=2}^{R-1} u_i\left(\hat{s}_i, s_{-i}^r\right)\left[\mu_i\left(s_{-i}^1, ..., s_{-i}^r\right) - \mu_i\left(s_{-i}^1, ..., s_{-i}^{r-1}\right)\right]$$
$$+\alpha_i u_i\left(\hat{s}_i, \underline{s}_{-i}\right)\left[1 - \mu_i\left(S_{-i} \backslash \underline{s}_{-i}\right)\right] + (1-\alpha_i)\, u_i(\hat{s}_i, \bar{s}_{-i})\, \bar{\mu}_i(\bar{s}_{-i})$$
$$+(1-\alpha_i) \textstyle\sum_{r=2}^{R-1} u_i\left(\hat{s}_i, s_{-i}^r\right)\left[\bar{\mu}_i\left(s_{-i}^1, ..., s_{-i}^r\right) - \bar{\mu}_i\left(s_{-i}^1, ..., s_{-i}^{r-1}\right)\right] + (1-\alpha_i)\, u_i\left(\hat{s}_i, \underline{s}_{-i}\right)\left[1 - \bar{\mu}_i\left(S \backslash \underline{s}_{-i}\right)\right]$$
$$= \alpha_i u_i(\hat{s}_i, \bar{s}_{-i})\, \mu_i(\bar{s}_{-i}) + \alpha_i \textstyle\sum_{r=2}^{R-1} u_i\left(\hat{s}_i, s_{-i}^r\right)\left[\mu_i\left(s_{-i}^1, ..., s_{-i}^r\right) - \mu_i\left(s_{-i}^1, ..., s_{-i}^{r-1}\right)\right]$$
$$+\alpha_i u_i\left(\hat{s}_i, \underline{s}_{-i}\right)\left[1 - \mu_i\left(S_{-i} \backslash \underline{s}_{-i}\right)\right] + (1-\alpha_i)\, u_i(\hat{s}_i, \bar{s}_{-i})\left[1 - \mu_i(S_{-i} \backslash \bar{s}_{-i})\right]$$
$$+(1-\alpha_i) \textstyle\sum_{r=2}^{R-1} u_i\left(\hat{s}_i, s_{-i}^r\right)\left[\mu_i\left(s_{-i}^r, ..., s_{-i}^R\right) - \mu_i\left(s_{-i}^{r+1}, ..., s_{-i}^R\right)\right] + (1-\alpha_i)\, u_i\left(\hat{s}_i, \underline{s}_{-i}\right) \mu_i\left(\underline{s}_{-i}\right).$$

Similarly if i plays strategy $\tilde{s}_i$ (s)he receives utility:

$$V_i(\tilde{s}_i) = \alpha_i u_i(\tilde{s}_i, \bar{s}_{-i})\, \mu_i(\bar{s}_{-i}) + \alpha_i \textstyle\sum_{r=2}^{R-1} u_i\left(\tilde{s}_i, \sigma_{-i}^r\right)\left[\mu_i\left(\sigma_{-i}^1, ..., \sigma_{-i}^r\right) - \mu_i\left(\sigma_{-i}^1, ..., \sigma_{-i}^{r-1}\right)\right]$$
$$+\alpha_i u_i\left(\tilde{s}_i, \underline{s}_{-i}\right)\left[1 - \mu_i\left(S_{-i} \backslash \underline{s}_{-i}\right)\right] + (1-\alpha_i)\, u_i(\tilde{s}_i, \bar{s}_{-i})\left[1 - \mu_i(S_{-i} \backslash \bar{s}_{-i})\right]$$
$$+(1-\alpha_i) \textstyle\sum_{r=2}^{R-1} u_i\left(\tilde{s}_i, \sigma_{-i}^r\right)\left[\mu_i\left(\sigma_{-i}^r, ..., \sigma_{-i}^R\right) - \mu_i\left(\sigma_{-i}^{r+1}, ..., \sigma_{-i}^R\right)\right] + (1-\alpha_i)\, u_i\left(\tilde{s}_i, \underline{s}_{-i}\right) \mu_i\left(\underline{s}_{-i}\right).$$

In the limit as $\bar{\gamma}$ tends to 1 all the terms involving μ_i tend to 0. Hence $V_i(\hat{s}_i)$ tends to $\alpha_i u_i\left(\hat{s}_i, \underline{s}_{-i}\right) + (1-\alpha_i)\, u_i(\hat{s}_i, \bar{s}_{-i})$ and $V_i(\tilde{s}_i)$ tends to $\alpha_i u_i\left(\tilde{s}_i, \underline{s}_{-i}\right) + (1-\alpha_i)\, u_i(\tilde{s}_i, \bar{s}_{-i})$. Since $\hat{s}_i \in \mathcal{A}, \tilde{s}_i \notin \mathcal{A}$, $\alpha_i u_i\left(\hat{s}_i, \underline{s}_{-i}\right) + (1-\alpha_i)\, u_i(\hat{s}_i, \bar{s}_{-i}) - \left[\alpha_i u_i\left(\tilde{s}_i, \underline{s}_{-i}\right) + (1-\alpha_i)\, u_i(\tilde{s}_i, \bar{s}_{-i})\right] > 0$. It follows that $\tilde{s}_i$ will not be played when $\bar{\gamma}$ is sufficiently high. ■

Proof of Proposition 4.1 By Lemma C.9, if the minimal degree of ambiguity is sufficiently high, $\operatorname{supp} \nu_i \subseteq \operatorname{argmax}_{s_i \in S_i} \left\{\alpha_i u_i\left(s_i, \underline{s}_{-i}\right) + (1-\alpha_i)\, u_i(s_i, \bar{s}_{-i})\right\}$, for $1 \leqslant i \leqslant n$. If α_i is also sufficiently high (resp. low) then $\operatorname{supp} \nu_i \subseteq \operatorname{argmax}_{s_i \in S_i} u_i\left(s_i, \underline{s}_{-i}\right)$ (resp. $\operatorname{supp} \nu_i \subseteq \operatorname{argmax}_{s_i \in S_i} u_i(s_i, \bar{s}_{-i})$). By Theorem 4.1 the resulting equilibrium is smaller (resp. greater) than the lowest (resp. highest) equilibrium without ambiguity. ■

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