Credit Risk in General Equilibrium

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This paper contributes to the literature on default in general equilibrium. Borrowing and lending takes place via a clearing house (bank) which monitors agents and enforces contracts. Our model develops a concept of bankruptcy equilibrium that is a direct generalization of the standard general equilibrium model with financial markets. Borrowers may default in equilibrium and returns on loans are determined endogenously. Restricted to a special form of mean variance preferences, we derive a version of the Capital Asset Pricing Model with bankruptcy. In this case we can characterize equilibrium prices and allocations and discuss implications for credit risk modeling.

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1 Introduction

This paper contributes to the literature about default in general equilibrium. It presents a framework that allows us to use the central ideas of this literature while at the same time keeping the model as close as possible to the standard general equilibrium model with financial markets. We also aim at providing a framework which allows a characterization of equilibrium prices and allocations going beyond abstract existence results and fully parametrized examples. We hope that this approach can mobilize the conceptual power of general equilibrium thinking for financial stability analysis.

Our paper makes the following new contributions: First, we model default penalties by extending the choices of agents to negative consumption plans. While the non-negative parts of the consumption plan is interpreted in the usual way the negative part of the consumption plan is identified with a default penalty. The utility function can simultaneously evaluate both consumption plans and default penalties. Second, we model default by restricting the default options of individuals to situations where they default only if all resources from endowments and other assets are insufficient to cover existing promises. Such a bankruptcy mechanism can be implemented while keeping anonymity of market exchange and price taking behavior. This is possible by introducing a clearing house that pools financial promises and has a monitoring and enforcement technology to verify bankruptcies and initiate the feasible payments. While building on the existing literature on default in general equilibrium our approach provides a structure that allows for a simple analysis of bankruptcy as an equilibrium phenomenon. It allows staying close to the standard model without default on financial promises. While non-linearities are introduced by driving a wedge between borrowing and lending rates, the single person decision problems remain linear. Looking at bankruptcies rather than arbitrary defaults, excludes equilibria of extreme pessimism, where financial markets break down. These potential pessimism equilibria led to relatively complicated equilibrium refinements in the previous literature. In our approach these refinements are not needed. We can thus give a fairly standard existence proof. Furthermore, making some more specific assumptions on preferences, our approach allows for an application that yields a bankruptcy version of the Capital Asset Pricing Model (CAPM). This bankruptcy-CAPM contains the standard CAPM as a special case. This allows for a more detailed study of the economics of bankruptcy and credit risk in general equilibrium by explicitly pinning down equilibrium prices and allocations.

Related Research The older literature on bankruptcy (Green [1973], Grandmont [1977] , Grandmont [1985]) is mostly conducted in a temporary equilibrium setting. It addresses already the main issues regarding existence of equilibrium. In particular it analyzed how to get a well-defined optimization problem of a borrower/lender. Issues that were already discussed in this literature are equilibrium existence with choice sets that are unbounded below, the modeling of penalties for choices which do not respect the feasibility of repayment in all states of the world ("planned default") as well as continuity of the budget and demand correspondence when the asset span depends on endogenous variables (Radner [1972]).

The literature on debt contracts during the eighties is focused primarily on information problems. An important topic in this literature was that costly state-verification can provide a justification for debt contracts with state-independent payoffs in states with no monitoring and where all assets of the debtor will be seized in states where no repayment occurs and monitoring takes place(Townsend [1979], Gale and Hellwig [1985]). This literature also discussed the idea that costly monitoring of debt contracts provides a rationale for a specialized institution that can be interpreted as a bank (Diamond [1984]).

The more recent literature most importantly Zame [1993] and Dubey et al. [2005] focuses on default rather than bankruptcy. This means that agents, regardless of their resources may decide to which degree they will fulfill financial promises, given a penalty proportional to the shortfall. The general equilibrium model with default and penalties was generalized to a continuum of states (Araujo et al. [1998]) and to infinite horizon models (Araujo et al. [1996]). [1]

While our model is close to the models used in this literature it has three main features in which it differs: First, we use a more general approach to model penalties. Instead of working with a separable function that adds utility of consumption and a penalty function for exceeding financial promises we use one function which simultaneously evaluates consumption and penalties. Second we study an "ability to pay" model (bankruptcy) rather than a "willingness to pay model" (default). In combination with our approach of modeling penalties, this feature allows an analysis that is closer to the standard model without default than the previous literature. Moreover, in special cases it also allows for a characterization of equilibrium prices and allocations. Finally the bankruptcy approach excludes no trade-equilibria due to extreme pessimism and therefore does not need equilibrium refinements as in Zame [1993] and Dubey et al. [2005].

We appeal to monitoring- and state-verification costs in order to justify the institution of a clearing house (bank) which acts as trading partner for debtors and lenders. It buys and sells bonds at differing rates for borrowing and lending and guarantees feasibility of contracts. It verifies the remaining assets of a debtor in case of bankruptcy and determines an aggregate recovery rate. In our model, individuals buy bonds from the clearing house at a guaranteed rate and lenders pay back to the clearing house either the contracted amount or their assets will be verified and seized.

Modeling the choice set of agents in a way that allows for feasible choices with negative components also arose in the finance literature discussing the notion of arbitrage (see Werner [1987], Dana et al. [1999]).[2] In contrast to our paper this comes from introducing assets and securities directly in the preference relation. A negative component in a

[1] There is a second important class of general equilibrium models allowing for default. Rather than modeling default consequences by penalties in this literature all financial promises have to be secured by collateral in the form of assets or durable consumption goods. When a default occurs collateral is transferred from the borrower to the lender. The key reference in this literature is Geanakoplos and Zame [2014].

[2] In addition to providing new results the paper by Dana et al. [1999]) contains an overview of this literature and clarifies all the different arbitrage notions used there. Insead of enumerating the rather long list of papers on arbitrage and general equilibrium we refer the interested reader to this paper.

choice vector of an agent is in this case to be interpreted as a short position in a security. Negative consumption plans are not analyzed or allowed for in this literature.

Finally we would like to discuss our paper in the context of the large literature on pecuniary default penalties in infinite horizon models, such as Kehoe and Levine [1993], Alvarez and Jerman [2001], Kocherlakota [2008], Hellwig and Lorenzoni [2009], Bloise and Reichlin [2011] and Azariadis and Kaas [2013]. All of these papers consider in some way or another the option of temporary or partial exclusion of agents from financial markets trading as a consequence of defaulting on financial promises. This default option creates additional constraints in the agents decision problem and impacts on the welfare properties of equilibria. Actual default does not occur in equilibrium. In our model default (bankruptcy) does occur in equilibrium.

Structure of the paper We begin in section (2) with the analysis of a simple example of competitive borrowing and lending that illustrates the main concepts and ideas of our analysis of credit risk in general equilibrium. In section (3) we describe and analyze the model, we define the concept of bankruptcy equilibrium. In section (4) we present the central results. We first prove existence of a bankruptcy equilibrium. In a next step, making more specific assumptions on preferences we derive a version of the Capital Asset Pricing Model with bankruptcy. This allows us to characterize equilibrium prices and quantities. Section (5) briefly discusses the implications of our analysis for credit risk modeling. Finally section(6) concludes. An appendix contains proofs of propositions stated in the main text.

2 Bankruptcy Equilibrium: An Example

We begin our analysis of bankruptcy equilibrium with an example that is simple enough to allow for a graphic exposition, yet rich enough to introduce the main concepts and ideas. The problem we would like to analyze is competitive markets for borrowing and lending with the possibility of bankruptcy. Building on the main ideas in the literature on default in general equilibrium, we aim at a conceptual framework that is both a simple and a natural extension of the traditional concept of financial market equilibrium as discussed for instance in Magill and Quinzii [1996].

In our example, two risk averse agents live for one period starting today ($t = 0$) and ending tomorrow ($t = 1$). They have endowments of a consumption good today and tomorrow. The endowments are described by the vectors $\omega^1 = (\omega_0^1, \omega_1^1)$ and $\omega^2 = (\omega_0^2, \omega_1^2)$ with all entries positive. They have standard preferences for the consumption good modeled by a utility function $u^i : X^i \mapsto \mathbb{R}$, where X^i denotes the agent's consumption set.

At $t = 0$, agents competitively trade a bond which promises one unit of income at $t = 1$. The bond trades at price q and the quantities of the bond chosen by the agents are denoted by z^i. The bond is in zero net supply. It is a financial instrument that allows borrowing and lending and, thus, for an inter-temporal transfer of the consumption good.

In the textbook model of financial market equilibrium, institutional arrangements of market exchange are such that, knowing only their own goals and endowments and observing prices, agents will always be able to stay within their budget constraints in each state of the world. We relax this assumption. Agents can make financial promises exceeding their resources tomorrow in some state of the world. Hence, bankruptcy may occur. We will consider uncertainty in the general form of our model described in the next section. In this example, however, for the sake of outlining our institutional setup in the simplest possible way, we restrict attention to a single state of the world at $t = 1$.

To allow for anonymous and competitive exchange the institutional and informational structure of financial market exchange is specified in the following way. There is a clearing house through which the agents can indirectly exchange financial promises. The clearing house is a passive intermediary. In $t = 0$, it collects payments from agents long in the bond. With these funds it provides resources to agents with a short position in the bond. In $t = 1$, the clearing house collects repayments. If there is a shortfall in contracted repayments the clearing house will confiscate the resources of the respective agents and use these proceeds to partially redeem financial claims of lenders. The distribution of proceeds follows a proportional rationing rule. The amount of the collected proceeds endogenously determines the rate of return for lenders. Moreover, if bankruptcy occurs, a utility penalty is applied to the agent who does not fulfill his promises. The penalty increases with the shortfall. Bankruptcy penalties are a modeling shortcut to describe costs of default for the agent without modeling them in detail.

We view this institutional arrangement as a costly state verification setup (Townsend [1979]), where the clearing house monitors, collects and distributes payments. It has the possibility to verify the state in case payments are not forthcoming and it has the authority to enforce payments and apply penalties.

A simple way to formalize these ideas is to extend the domain X^i of preferences: We assume that agents have a utility function with standard properties allowing to evaluate *both* positive consumption plans as well as penalties. Penalties are identified with negative consumption plans. Thus, with bankruptcy penalties, $X^i \subset \mathbb{R}_+ \times \mathbb{R}$, in contrast to the no bankruptcy case $X^i \subset \mathbb{R}_+ \times \mathbb{R}_+$. We will assume that a negative consumption value, $x^i < 0$, reflects the degree of bankruptcy, while a non-negative consumption value, $x^i \geq 0$, represents actual consumption. With this interpretation, our utility function embeds a *penalty function* which is strictly increasing in the value of the planned shortfall in a bankruptcy: [3] $x^{i-} := x^i \wedge 0$. Denoting the value of actual consumption by the non-negative value $x^{i+} = x^i \vee 0$ we have $x_1^i = x_1^{i+} + x_1^{i-}$. Thus, the consumer evaluates a consumption plan with respect to *both* the real consumption x_s^{i+} and the default penalty x_s^{i-} by a common utility index $u^i(x_0^i, x_1^i) = u^i(x_0^i, (x_1^{i+} + x_1^{i-}))$. To our knowledge, this is a novel approach to model utility penalties for bankruptcy that has not been used in the literature before.

The general idea to model costs of default by a utility penalty is not new and may be traced back to Dubey et al. [2005] and Zame [1993]. As most of the older literature, Dubey et al. [2005] and Zame [1993] model utility from real consumption $\widetilde{u}^i(x_0^i, x_1^{i+})$ and

[3] We use the notation $\vee$ and $\wedge$ as the maximum and minimum operator. Applied to vectors the operators give the component-wise maximum of minimum.

the penalty function $w^i(x_1^{i-})$ as additively separable preferences $\widetilde{u}^i(x_0^i, x_1^{i+}) + \lambda w^i(x_1^{i-})$. This utility function can be viewed a a special case of our utility function which evaluates real consumption and default penalty jointly: $u^i(x_0^i, x_1^i) := u^i(x_0^i, (x_1^{i+} + x_1^{i-}))$. Another example consistent with our modeling approach would be the quadratic utility function as observed by Magill and Quinzii [2000] [4].

There are thus two key elements in our bankruptcy model which in combination distinguish it from the older literature: First agents can not arbitrarily default on their promises. As long as their resources allow they have to pay. In our institutional framework this can be enforced because the clearing house is assumed to have the monitoring technology to enforce feasible payments. Second the utility function can rank bundles of consumption plans and bankruptcy penalties thought of as negative consumption plans. Since exchange of promises is intermediated by the clearing house this exchange is anonymous. Agents only observe security prices and the repayment rate on the bond.

This institutional framework can be viewed as a standard debt contract as analyzed in Gale and Hellwig [1985]: A standard debt contract is characterized by fixed repayment in all states where no bankruptcy occurs and full recovery (seizure of the entire endowment) in bankruptcy states.

For the simple example with two agents, one state and one bond, the equilibrium problem can be discussed geometrically in a net trade diagram as in [Magill and Quinzii, 1996, Figure 10.1.]. The diagram is drawn in the space of net income transfers defined by $\tau^i = x^i - \omega^i$, with net income transfers in $t = 0$ on the x-axes and net income transfers in $t = 1$ on the y-axes. With the possibility of bankruptcy, the payoff of a bond depends on whether the agent is a borrower or a lender. Hence, long and short positions have to be distinguished. At price q an agent can achieve all net income transfers along the ray generated by the vector $(-q, r)$ with a long position $z_+^i = (0 \vee z^i)$. With a short position $z_-^i = -(0 \wedge z^i)$ he can achieve all net income transfers starting at the origin and extending along the ray generated by the vector $(q, -1)$. The difference in the return rate on bond holdings and on bond sales reflects the fact that in case of a bankruptcy the bond pays not the promise 1 but only the smaller return rate r

$$r = \frac{z_-^i \wedge \omega_1^i}{z_-^i}. \tag{1}$$

If we denote by $Z = \mathbb{R}_+ \times \mathbb{R}_+$ the set of feasible net trades in portfolios $z^i = (z_+^i, z_-^i)$ and by T the bond return matrix

$$T = \begin{bmatrix} -q & q \\ r & -1 \end{bmatrix},$$

the set of feasible income transfers is given by

$$\mathcal{C} = \{\tau \in \mathbb{R}^2 | \tau = Tz \quad z \in Z\}$$

[4]Assume there are $S+1$ states of the world, each state occurring with probability $\rho_s > 0$, this would be the function $u^i(x^i) = -\frac{1}{2}\sum_{s=0}^{S} \rho_s(\alpha^i - x_s^i)^2$. If we had a penalty function that gives a huge disutility to zero consumption, such as the logarithmic Bernoulli utiliy function, we would be back in the standard "no-bankruptcy" model.

where the index of agents has been suppressed.

Without bankruptcy (the case $r = 1$) this set is a linear space since the promise of the bond is always 1 no matter whether the position is long or short. This linear space becomes the cone $\mathcal{C}$ in the bankruptcy case.

The rays of the net transfer space $\mathcal{C}$ are drawn in the upper-left orthant and in the lower right orthant. This reflects the fact that financial markets must be free of arbitrage opportunities in equilibrium. So every unit long in the bond yielding a positive payoff r tomorrow requires to give up $-q$ in terms of consumption good today. Every unit short in the bond pays q units of the consumption good today but carries an obligation of repayment tomorrow and eventually a utility penalty. The requirement of no arbitrage is equivalent to the requirement that there exist strictly positive state prices $\pi = (\pi_0, \pi_1)$ in the polar cone of the net transfer cone $\mathcal{C}$:

$$\mathcal{C}^* = \{\pi \in \mathbb{R}^2 | \pi\tau \leq 0 \quad \forall \tau \in \mathcal{C}\} \tag{2}$$

In the graph this is the shaded cone starting at the origin in the upper right orthant of the net transfer space. In a model without bankruptcy this cone is a linear space. Since long bonds have a different return than short bonds, however, the present value of an income stream is different depending on whether it implies a positive or negative transfer at $t = 1$. Hence, the net transfer space has a kink. Bankruptcy leads to a non-linear equilibrium valuation of contingent claims. In Figure 1 it is shown that in equilibrium contingent claims that amount to a long position in the bond are valued at $\bar{\pi}^l$ while contingent claims that amount to a short position are valued at $\bar{\pi}^s$.

The consumption spaces of agents 1 and 2 in the net trade diagram are half spaces containing the endowment points and all bundles x^i to their right. To mark the difference to the textbook model the shadings of the axes are such that both negative as well as positive consumption plans belong to the choice sets of agents.

In this example, agent 2 is the lender. He finds it optimal to choose a consumption bundle which requires an investment in the bond. At his optimal decision the gradient of his utility function is orthogonal to the ray generated by the vector $(-q, r)$. Since negative consumption, interpreted as utility penalties, is possible, budget sets are not bounded below because the choice sets are not bounded from below[5]. Agent 1 is the borrower. His optimal choice is a bundle with high consumption today at the cost of no consumption plus a penalty tomorrow. At the optimum the gradient of his utility function is orthogonal to the ray generated by $(q, -1)$. Borrowing excessively today, he can achieve this choice within the prevailing financial structure.

We define a bankruptcy equilibrium as a situation where agents take optimal decisions, financial markets clear for these decisions and positive consumption plans are compatible with the available resources. The picture illustrates how the clearing house arrangement makes such an equilibrium allocation feasible. Contrary to the standard model in which

[5] To obtain a bounded budget set additional assumptions are necessary. As explained in the next section, one can either introduce a lower bound bond trades through some kind of short selling restriction as in Radner [1972] or appeal to a stricter notion of no arbitrage as in Werner [1987] and Dana et al. [1999].

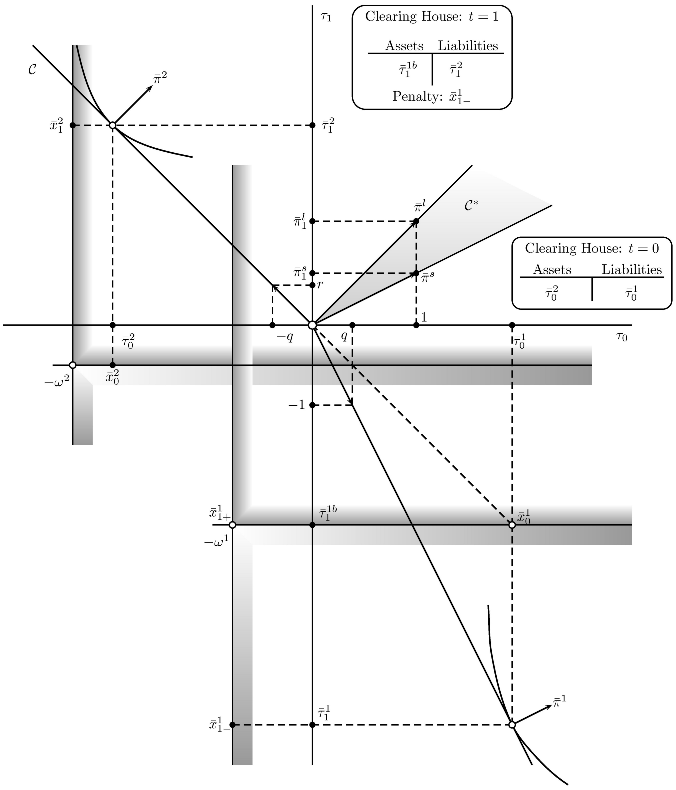

Figure 1: Bankruptcy equilibrium for the case with two agents and one state of the world. Agent 1 is the borrower and agent 2 is the lender. Agent 1 takes an optimal choice by going bankrupt at $t = 1$. In equilibrium $(\bar{q}, \bar{r})$ are such that the actual net trades balance. The bond is exchanged indirectly via a clearing house which has the opportunity to enforce payments and apply penalties and rationing of claims in a bankruptcy. The clearing house is like a passive intermediary. It's balance sheet at $t = 0$ and at $t = 1$ are shown in the picture. The balance sheet shows that the claim of agent 2, the lender is rationed to $\bar{\tau}_1^{1b}$ and the clearing house confiscates and distributes agent 1's, the borrower's, endowment and applies the utility penalty $\bar{x}_-^1$.

q adjusts such that the net transfer space is rotated until all net trades balance, in a bankruptcy equilibrium two parameters, bond price q and the recovery rate r, must be adjusted.

Denoting the agents' net transfers for given (q, r) by $\tau^1(q, r))$ and $\tau^2(q, r)$, the bond price and return rate $(\bar{q}, \bar{r})$ are a *bankruptcy equilibrium* when

$$(\tau^1(\bar{q}, \bar{r})) \vee -\omega^1) + (\tau^2(\bar{q}, \bar{r}) \vee -\omega^2) = 0.$$

To interpret the equilibrium condition in terms of portfolio choices, let $z^1(q, r)$ and $z^2(q, r)$ be the optimal portfolio choices and $x^1(q, r)$ and $x^2(q, r)$ the optimal consumption choices at (q, r). Since agents must not choose a negative consumption in $t = 0$, in equilibrium, security markets must clear [6],

$$z^1(\bar{q},\bar{r}) + z^2(\bar{q},\bar{r}) = 0,$$

and consumption in $t = 1$ must be feasible,

$$(x^1(\bar{q},\bar{r}) \vee 0) + (x^2(\bar{q},\bar{r}) \vee 0) = \omega_1^1 + \omega_1^2.$$

The condition that consumption at $t = 1$ must be feasible is equivalent to rational expectations about the return rate $\bar{r}$, i.e., the return rate that enters consumers' decision problems in equilibrium equals the return rate actually realized at $t = 1$. Hence, an alternative set of equilibrium conditions would require security markets to clear and agents correctly expecting the equilibrium return rate $\bar{r}$.[7]

The diagram shows how the bond market can equilibrate with bankruptcy. The transfer space of the lender has to be rotated by rationing his claim from the ray $(-q, 1)$ to $(-q, r)$ such that if this ray is prolonged to an imaginary linear space it passes through the zero consumption line of agent 1 at $\bar{x}_1^1 = 0$. In this position the market can clear because at this point the *actual* net income transfers sum to zero and thus $(\tau^1(\bar{q},\bar{r})) \vee -\omega^1) + (\tau^2(\bar{q},\bar{r}) \vee -\omega^2) = 0$. The picture also shows the balance sheets of the clearing house at $t = 0$ and at $t = 1$.

This example reveals in a nutshell the basic ideas and concepts of our notion of a bankruptcy equilibrium. Clearly, in a two-agent example, the clearing-house mechanism appears artificial. For many agents, however, trading financial promises solves a complicated information and coordination problem. Moreover, there are securitization markets which work exactly like asset pools financed by debt as in the clearing house construction.

Finally, how is credit risk entering the picture? This aspect of bankruptcy can only be seen in the general more complex model with many states of nature which we will develop in the next section. In the context of several states of nature, an endowment of zero in a single state would make loans impossible and, thus, eliminate all intertemporal trade, if no bankruptcy were possible. In such a situation, a bankruptcy equilibrium can be an improvement for all agents even in the face of bankruptcy penalties. This aspect of default was a central point of Zame [1993] as well as Dubey et al. [2005].

The general model considers also other financial instruments which may be used for risk sharing. Agents then take optimal portfolio decisions and financial instruments are priced according to their risk characteristics. Clearly the decision of some agents to choose a consumption plan that implies bankruptcy in some state is a credit risk from the viewpoint of the lenders. This is a risk, however, that arises endogenously as a consequence of the agents' decisions. Hence, parameters of credit risk such as the probability of default, the exposure at default and the recovery rate, which are assumed to follow an exogenous probability law in the usual credit risk models, will be endogenously determined in equilibrium.

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[6] This condition is equivalent to $\tau_0^1 + \tau_0^2 = 0$.

[7] This is analogous to the general equilibrium, multigood financial market model, where agents have to correctly anticipate equilibrium goods prices at $t = 1$ when making their plans today (see Radner [1972]).

3 The Model

3.1 A Bond Equity Economy

We consider a pure exchange economy with one commodity and a finite number I of agents. There are two dates, $t = 0$ and $t = 1$, and a finite number S of states of the world at date $t = 1$.

Each agent is characterized by a closed and convex choice set $X^i \subset \mathbb{R}_+ \times \mathbb{R}^S$, a continuous, strictly increasing and concave utility function $u^i : X^i \to \mathbb{R}$ [8] and an initial endowment $\omega^i \in X^i \cap \mathbb{R}^{S+1}_{++}$. We denote by ω^i_0 the endowment at $t = 0$ and by $\omega^i_{\mathbf{1}} = (\omega^i_1, \ldots, \omega^i_S)$ the endowment vector at $t = 1$. In a similar manner we denote by $x^i = (x^i_0, x^i_{\mathbf{1}}) \in X^i$ a consumption-bankruptcy plan of agent i.

The choice set is the same for all agents and consists of both consumption plans and potential bankruptcy penalties at $t = 1$. We assume that for all agents $X^i \subset \mathbb{R}_+ \times \mathbb{R}^S$, since we identify negative consumption plans with bankruptcy penalties. There are no outstanding claims in period 0. Hence, we assume that $x^i_0 \geq 0$. As explained above, the utility function u^i evaluates *both* consumption plans and bankruptcy penalties.

To achieve a consumption profile optimally adapted to their risk preferences agents can trade $J + 1$ securities. These securities are best thought of as a bond-equity security structure as in [Magill and Quinzii, 1996, p. 177]. First, there are J securities with payoff profile $y^j = (y^j_1, \ldots, y^j_S) \in \mathbb{R}^S_+$. These state-contingent payoffs can be though of as the exogenous output of J firms at $t = 1$. Equity represents claims to this output. Each agent owns $\delta^i \in \mathbb{R}^J_{++}$ claims initially. Equity can not be sold short. The $S \times J$ matrix of all equity payoffs y^j is denoted by Y. Each consumer chooses a portfolio $\theta^i \in \mathbb{R}^J_+$ of equity. The net-purchases (sales) of equities are denoted by $z^i_e = (\theta^i - \delta^i) \in \mathbb{R}^J_+ - \{\delta^i\}$. The no-short sale constraint for equities requires that z^i_e must always be greater or equal to $-\delta^i$. Equity prices are denoted by $q_e \in \mathbb{R}^J$.

In addition to the equity markets there is also a market for a debt instrument. It can be thought of as a bond, which allows the agents to make loans. For simplicity, we consider only one bond. The bond promises one unit of the consumption good in each state of the world. Agents can take on debts by trading the bond. A consumer may sell bonds even if the sale would require repayments exceeding the resources in some state. The consumer may well choose a bond sale leading to "negative consumption" in some state if the benefits from the loan in other states justify it. As discussed before, we interpret the disutility from "negative consumption" as a bankruptcy penalty which the agent suffers, even if the actual allocation will deliver a feasible consumption of zero in such a state.

Trade in bonds takes place between consumers and an agency which operates as a clearing house managing repayments. If there is a state where an agent's endowment does not suffice to cover the promised repayment from a bond sale, the agent will be bankrupt. A bankrupt agent will lose all resources in the respective state to the clearing house agency. The agency determines a return rate $r_{\mathbf{1}} \in (0, 1]^S$ which is paid to the bond holders. Hence, an agent who has invested in the bond has to take into account

[8] Strictly increasing means that for any $x, x' \in \mathbb{R}^n_+$ with $x \geq x'$ and $x \neq x'$, $u^i(x) > u^i(x')$.

that the payoff profile of a bond purchase is perhaps only $r_{\mathbf{1}} \in (0,1]^S$, falling short of the contracted repayment in some states.

We use a different notation for long and short positions in the bond. We define the positions of agent i long in the bond by z^i_{b+} and the positions short in the bond by z^i_{b-}. We assume that there is a short-selling constraint κ on the bond. The set of feasible net trades in portfolios is the set $Z := \mathbb{R}_+ \times [0,\kappa] \times \left(\mathbb{R}^J_+ - \left\{\delta^i\right\}\right)$. A portfolio is a tuple $z^i = (z^i_{b+}, z^i_{b-}, z^i_e) \in Z$. We assume that $J < S$. Hence, our analysis covers both the case of incomplete as well as complete markets. If $J = S - 1$ the availability of a bond completes the financial market system. When $J < S - 1$ we have incomplete markets.

Normalizing the price of the consumption good to 1, we can now define the budget set of agent i by

$$\mathbb{B}^i(q, r_{\mathbf{1}}) = \left\{ (x^i_0, x^i_{\mathbf{1}}) \in X^i \;\middle|\; \begin{array}{lcl} x^i_0 - \omega^i_0 & \leq & -q_b z^i_{b+} + q_b z^i_{b-} - q_e z^i_e \\ x^i_{\mathbf{1}} - \omega^i_{\mathbf{1}} - Y\delta^i & \leq & r_{\mathbf{1}} z^i_{b+} - \mathbb{1} z^i_{b-} + Y z^i_e \\ & & (z^i_{b+}, z^i_{b-}, z^i_e) \in Z \end{array} \right\} \tag{3}$$

Note that the recovery rate r_s on the bond in each state is taken as a parameter by the consumer. Consumers are assumed to maximize their utility subject to this budget constraint. Recall also that consumption x^i_s may become negative in some state s, indicating that the consumer is bankrupt in this state and receives a bankruptcy penalty corresponding to this negative consumption value.

The bounds on short sales implicitly bound the choice set X^i from below [9]. This is a modeling assumption that ensures that we always have a well defined agent optimization problem when security prices fulfill a standard no arbitrage condition as for instance in Magill and Quinzii [1996] . An advantage of this lower bound is the fact, that we can consider various special cases depending on the size of this borrowing constraint κ. For $\kappa = 0$, we cover the case of an economy without credit. For $\kappa > 0$ but small enough, one can consider an economy without bankruptcy. With $\kappa > 0$ sufficiently large, bankruptcy is a possibility and returns on the bond will be determined endogenously.

3.2 Bankruptcy and the clearing house

Bankruptcy refers to a set of institutional arrangements specifying the reallocation of claims among economic agents. An agent is *bankrupt* , when the value of his debts exceeds the value of his assets. In the two period framework employed here this condition can be unambiguously defined.[10]

[9] There is a literature starting from Werner [1987] and analyzed in depth in Dana et al. [1999] where stronger no-arbitrage notions together with additional assumptions on utility functions endogenously bound the choice set, so that absence of arbitrage is necessary and sufficient for equilibrium existence when the choice set is unbounded. Since we want to focus attention on the bankruptcy clearing mechanism and the implications for the endogenous return on bonds we choose the short selling constraint approach.

[10] Such a formalization has been used in the literature in different versions by Modica et al. [1998], Sabarwal [2003], Araujo and Pascoa [2002]. It is also close to the framework of Eisenberg and Noe [2001], which shows how one can extend our bankruptcy rule to many loan instruments and non-anonymous bankruptcy in a pure balance sheet mechanics framework. A bankruptcy occurs if agents

In case of a bankruptcy, all remaining assets of the debtor will be seized and distributed among the creditors. The remaining debt will be forgiven. Two institutional aspects are essential for the economic outcome: Firstly, how will the remaining assets be distributed among the claim holders? Secondly, what kind of penalty will be imposed on the bankrupt agent for not paying back the contracted amount of debt?

Bankruptcy laws specify these rules. Penalties have the function of influencing a borrower's choice of the debt level at $t = 0$. Penalties for bankruptcies usually consist in excluding bankrupt individuals, at least temporarily, from further credit and constraining their consumption to a minimum for some period. These penalties depend also on the size of the losses which creditors suffer. Modeling the consequences of bankruptcy by a utility penalty is a simplification for not modeling these consequences in detail.

If a bankruptcy occurs existing nominal claims of the bond holders can no longer be satisfied. In order to model bankruptcy in perfect competition, where agents act as price takers, the bankruptcy mechanism must be anonymous. Anonymity of bankruptcy can be formalized by assuming that bond transactions are mediated through some central clearing institution that distributes shortfalls on promised payments among creditors. In this model, the clearing house collects the remaining assets of bankrupt agents $\omega_s^i + Y_s(\delta^i + z_e^i) < z_{b-}^i$ and distributes their value to the creditors. If repayments $\left(\sum_{i=1}^I z_{b-}^i \wedge (\omega_s^i + Y_s(\delta^i + z_e^i)\right)$ fall short of aggregate promises $\sum_{i=1}^I z_{b-}^i$ in some state s then these claims will be reduced proportionally.[11] Hence, the *return rate* [12] $r_s \in [0, 1]$, is

$$r_s = \begin{cases} \frac{\sum_{i=1}^I z_{b-}^i \wedge (\omega_s^i + Y_s(\delta^i + z_e^i))}{\sum_{i=1}^I z_{b-}^i} & \text{if } \sum_{i=1}^I z_{b-}^i > 0 \\ 1 & \text{if } \sum_{i=1}^I z_{b-}^i = 0 \end{cases} \tag{4}$$

When planning their consumption and investments consumers will take this *recovery rate* r_s into account as an expected parameter which *will be determined in equilibrium.*

3.3 Bankruptcy Equilibrium

Let $u = (u^1, \ldots, u^I)$, $\omega = (\omega^1, \ldots, \omega^I)$ and $\delta = (\delta^1, \ldots, \delta^I)$ be the vectors of individual utility functions and individual endowments of goods and equity, respectively, and denote by $V = [-\mathbb{1}, Y]$ the exogenously given equity payoffs. We denote by $\mathcal{E} = (u, \omega, \delta, V)$ the corresponding economy.

cannot repay their due liabilities. In contrast to Zame [1993] and Dubey et al. [2005], we do not allow agents to strategically default on their loans. Agents will repay their debts as long as the value of their endowments and equity allows it. If liabilities exceed this value a bankruptcy occurs.

[11] Define for any two vectors $x, y \in \mathbb{R}^n$ the lattice operations $x \wedge y := (min(x_1, y_1), \cdots, min(x_n, y_n))$ and $x \vee y := (max(x_1, y_1), \cdots, max(x_n, y_n))$. By Y_s we denote the s-th row of the matrix Y

[12] Since the recovery rate is only defined when there is some trade in the bond we define the recovery rate in cases where there is no bond trade as 1 by a continuous extension.

Definition 1 (Equilibrium). *A* financial market equilibrium with bankruptcy *of the economy* $\mathcal{E} = (u, \omega, \delta, V)$ *is a tuple of consumption plans, portfolio choices, security prices and recovery rates* $(\bar{x}, \bar{z}_{b+}, \bar{z}_{b-}, \bar{z}_e, \bar{q}, \bar{r}_{\mathbf{1}}) \in X^I \times Z^I \times \mathbb{R}_+^{J+1} \times (0,1]^S$ *such that for all* $i = 1, \ldots, I$

(i) $\bar{x}^i \in \arg\max\{u^i(x^i) \mid x^i \in \mathbb{B}^i(\bar{q}, \bar{r}_{\mathbf{1}})\}$, (optimal behavior)

(ii) $\sum_{i=1}^I \bar{z}_{b+}^i - \sum_{i=1}^I \bar{z}_{b-}^i = 0$ *and* $\sum_{i=1}^I \bar{z}_e^i = 0$, (asset market clearing)

(iii) $\sum_{i=1}^I \bar{x}^{i+} = \sum_{i=1}^I (\omega^i + Y\delta^i)$, (feasible allocation)

(iv) $\bar{r}_s = \begin{cases} \frac{\sum_{i=1}^I (\bar{z}_{b-}^i \wedge (\omega_s^i + Y_s(\delta^i + \bar{z}_e^i))}{\sum_{i=1}^I \bar{z}_{b-}^i} & \textit{if } \sum_{i=1}^I \bar{z}_{b-}^i > 0 \\ 1 & \textit{if } \sum_{i=1}^I \bar{z}_{b-}^i = 0 \end{cases}$ (clearing mechanism)

The definition of equilibrium has some redundancy, since conditions (ii), (iii) and (iv) are not independent. Indeed, in the appendix we use this fact in the existence proof. The credit market clearing mechanism is a central feature of our economy. Including it explicitly in the definition of an equilibrium makes the interrelation between the clearing mechanism, the feasibility of consumption and the endogeneity of the return rate more transparent.

In equilibrium feasibility of the consumption allocation is guaranteed by condition (iii). This does not preclude that consumers choose an amount of debt which leads to a negative value of the consumption plan in some state. Hence, $\bar{x}_s^i$ can be negative in some states, representing the bankruptcy penalty experienced by this consumer. Creditors hold rational expectations about the recovery rate in states where bankruptcy occurs (iv). Only in this case security market clearing (*ii*) and good market clearing (*iii*) can be fulfilled simultaneously. Bankruptcy is factored into the asset price system $\bar{q}$. Note that the standard general equilibrium concept without bankruptcy is a special case of the financial market equilibrium with bankruptcy when $r_s = 1$ for all states s. This special case can be obtained by choosing the credit constraint κ small enough.

4 Bankruptcy Equilibrium: Results

4.1 Bankruptcy Equilibrium: Existence

We first show that our equilibrium concept is well defined. We show that, given the assumptions on preferences, endowments and securities in Proposition 1, a bankruptcy equilibrium will always exist.

Proposition 1 (Existence of Bankruptcy Equilibrium)**.** *Let $\mathcal{E}(u, \omega, \delta, V)$ be a finance economy. If*

A1	(consumption sets)	X^i *is a closed, convex and non-empty subset of* $\mathbb{R}^{S+1}$,
A2	(preferences)	$u^i : X^i \to \mathbb{R}$ *is continuous, strictly increasing, and strictly concave,*
A3	(endowments)	$\omega^i = (\omega_0^i, \omega_{\mathbf{1}}^i) \in X^i \cap \mathbb{R}_{++}^{S+1}$ *and* $\delta^i \in \mathbb{R}_{++}^J$,
A4	(asset returns)	*the matrix* $V = [-\mathbb{1}\ Y]$ *has full column rank,*
A5	(asset trades)	*there is* $\kappa > 0$ *such that* $z_{b-}^i \leq \kappa$ *for all* $i = 1, ..., I$.

then a bankruptcy equilibrium exists.

Proof: The proof is given in the appendix. □

Existence of a bankruptcy equilibrium has been proved in different settings by Sabarwal [2003] and for a different version of the model by Modica et al. [1998] and Araujo and Pascoa [2002].[13] Zame [1993] and Dubey et al. [2005] prove existence for a similar model with default. In these models consumers can deliberately decide what fraction of their promise they are going to repay. Hence, returns on a loan may become zero and there can be equilibria where expectations on bond recoveries are so pessimistic that there is no trade in the bond. In contrast, in our model where bankruptcy of an agent implies the seizure of all remaining assets by the clearing house, there is always some positive return on a loan from the endowments of the debtor. Hence, no trade in the bond due to overly pessimistic expectations cannot occur in an equilibrium under bankruptcy. While all the existence proofs in the all of the papers cited above are quite involved, our concept of a bankruptcy equilibrium allows the use of mostly standard arguments.

4.2 Bankruptcy Equilibrium and the CAPM

To make the bankruptcy model more useful for economic analysis, we want to go beyond the abstract discussion of the previous sections and add enough structure to the model such that we are able to study equilibrium prices and allocations explicitly. The aim is thus to arrive at a formulation of the model that is more specific than the abstract discussion yet more general than a fully parametrized example.

The formulation we are going to suggest and analyze now is to specify the bankruptcy model along the lines of a CAPM model, widely used in finance and economics.

[13] Araujo and Pascoa [2002] have no utility penalties but short selling constraints on the debt instruments, Sabarwal [2003] has T periods and no penalties but short selling constraints on the debt instruments. Modica et al. [1998] study a model where agents can become bankrupt without penalty in states of the world of which they are ex ante unaware of. Obviously all these models are closely related.

Instead of general preferences we now assume that preferences are defined by an additively separable, linear quadratic utility function

$$u^i(x^i) = \alpha_0^i x_0^i - \frac{1}{2}\sum_{s=1}^{S} \rho_s(\alpha_1^i - x_s^i)^2, \quad i = 1, \dots, I, \tag{5}$$

where we have also time 1 state probabilities are given by the ρ_s for $s = 1, \dots, S$. Note that this utility function provides a natural example of the class of preferences we studied in the previous section. It jointly evaluates the utility from consumption and utility penalties identified with negative consumption plans. This is an observation that has been made by Magill and Quinzii [2000] to give an interpretation to negative consumption plans usually admitted in the CAPM.[14]

Define by $\alpha_0 = \sum_i \alpha_0^i$, $\alpha_{\mathbf{1}} = \sum_i \alpha_1^i$ the aggregate preference parameters and by $\omega_{\mathbf{1}} = \sum_i \omega_{\mathbf{1}}^i$, $\delta = \sum_i \delta^i$ and $K = |I|\kappa$ the aggregate endowments of commodities, stocks and maximal loans. By assuming that all preference parameters $(\alpha_0^i, \alpha_1^i) \in \mathbb{R}^2_{++}$ are chosen such that for each agent $\alpha_1^i \mathbb{1} - \omega_{\mathbf{1}} - Y\delta - K \in \mathbb{R}^S_{++}$, where $\mathbb{1}$ is the S-dimensional vector consisting of components equal to 1, utility functions will be strictly increasing on the set of feasible state-contingent consumption.

Such an assumption is used for instance in Nielsen [1989] to ensure existence of equilibrium with preferences allowing satiation. Basically the assumption ensures monotonicity of utility on those regions of the choice set X that correspond to a feasible allocation.

Our specification of preferences assumes agents who care about the mean and the variance of their consumption plans. The advantage of this specific restriction is that it allows a characterization of equilibrium prices, portfolios and consumption.

We are now going to discuss equilibrium pricing and allocations in this CAPM specification of the model with bankruptcy. We discuss these results along the lines of the exposition in [Magill and Quinzii, 1996, chapter 3, 17] to highlight the similarity as well as the differences to the standard CAPM model.

4.3 Equilibrium Security Pricing: Adjusting the Market Portfolio for Credit Risk

Our first result refers to security pricing with linear quadratic preferences. To simplify the exposition we assume that the short selling constraints on equity will not be binding in equilibrium.

[14] Unlike in the general case discussed before these preferences have a satiation point and thus the utility function is not strictly increasing on its whole domain. It is well-known that one can obtain monotonicity on the relevant compact and convex set of state-contingent consumption by choosing for instance for all consumers satiation points outside the set of feasible allocations.

For our discussion we use the probability induced inner products

$$\langle x_{\mathbf{1}}, y_{\mathbf{1}} \rangle = \sum_{s=1}^{S} \rho_s x_s y_s \quad \forall\, x_{\mathbf{1}}, y_{\mathbf{1}} \in \mathbb{R}^S \text{ and} \tag{6}$$

$$\langle x, y \rangle = \sum_{s=0}^{S} \rho_s x_s y_s \quad \forall\, x, y \in \mathbb{R}^{S+1}, \tag{7}$$

where $\rho_0 := 1$.

Define the vector

$$\bar{\gamma} := \sum_{i=1}^{I} \nabla u^i(\bar{x}^i) = (\alpha_0, \alpha_1 \mathbb{1} - ((\omega_{\mathbf{1}} + Y\delta) - d_{\mathbf{1}}))^T,$$

where $d_{\mathbf{1}} = \sum_{i=1}^{I} d_{\mathbf{1}}^i := \sum_{i=1}^{I} (\mathbb{1} - r_{\ \mathbf{1}}) z_{b-}^i$ is the aggregate shortfall from promises on the bond. This vector $\bar{\gamma} = (\bar{\gamma}_0, \bar{\gamma}_{\mathbf{1}})$ expresses the equilibrium marginal evaluation of income streams that can be generated within the given financial structure.[15]

Proposition 2. *If $(\bar{x}, \bar{z}, \bar{q}, \bar{r}_{\mathbf{1}})$ is a bankruptcy equilibrium of the economy $\mathcal{E}(u, \omega, \delta, V)$ with non-binding short selling constraints on equity and strictly positive date zero consumption $\bar{x}_0^i$ then*

(i) $\bar{\gamma}$ fulfills the equation

$$\bar{\gamma} = (\alpha_0, \alpha_1 \mathbb{1} - \tilde{\omega}_{\mathbf{1}}),$$

(ii) denote the equilibrium market value of any income stream m that can be generated by a linear combination of the existing securities by $c(m)$. It fulfills the weak inequality

$$c(m) \geq \left\langle \frac{\bar{\gamma}_{\mathbf{1}}}{\alpha_0}, m \right\rangle = E\left(\frac{\bar{\gamma}_{\mathbf{1}}}{\alpha_0}\right) E(m) - \frac{1}{\alpha_0} cov(\tilde{\omega}_{\mathbf{1}}, m),$$

where $\tilde{\omega}_{\mathbf{1}} = \big((\omega_{\mathbf{1}} + Y\delta) - (\mathbb{1} - r_{\mathbf{1}}) \sum_i \bar{z}_-^i\big)$ is the aggregate endowment $\omega_{\mathbf{1}} + Y\delta$ reduced by the aggregate shortfall in promises on the bond market.

Proof: The proof is given in the appendix. □

From Proposition 2 we see that security prices in a bankruptcy equilibrium look similar to security prices in a financial market equilibrium without bankruptcy and quadratic preferences (see for instance [Magill and Quinzii, 1997, Proposition 1]. The most important difference is that in a bankruptcy equilibrium the role taken by the aggregate endowment $\omega_{\mathbf{1}}$ is now replaced by the aggregate endowment corrected for the aggregate shortfalls from bankruptcy $\tilde{\omega}_{\mathbf{1}}$.

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[15] Since we are using the probability inner product, the definition of $\nabla u^i(\bar{x}^i)$ is

$$\nabla u^i(\bar{x}^i) = \left(\frac{\partial u^i}{\partial x_0}(\bar{x}_0^i), \left(\frac{1}{\rho_s} \frac{\partial u^i}{\partial x_s}(\bar{x}_s^i) \right)_{s \in S} \right)$$

The pricing formula that results from the CAPM without bankruptcy shows that the price of a security is a decreasing linear function of the covariance of the income stream provided by this security with aggregate income risk in the economy as a whole. If for instance $\omega_{\mathbf{1}} \in \operatorname{span}(Y)$, the aggregate income risk could be interpreted as a benchmark portfolio, called the market portfolio in the finance literature. If a security is positively (negatively) correlated with aggregate income it's covariance value is negative (positive). Bankruptcy changes this insight in the sense that aggregate income risk can only be described in equilibrium. Aggregate income risk or the "market portfolio" has to be adjusted by the planned shortfall in financial promises. Aggregate income risk that is relevant for determining the covariance value of securities is endogenous.

Since this quantity determines the marginal value of income that can be achieved by trading in financial securities, the prices of *all* securities whether or not they are affected by credit risk are influenced by the trading of a defaultable security. This is of course a typical general equilibrium effect. The specific structure of our model allows however to say much more. The bankruptcy CAPM says that in the valuation of any financial security in a market containing at least one defaultable financial instrument, we can make a CAPM like valuation by correcting the market portfolio by the aggregate shortfall in promises on the credit instruments.

Since in a bankruptcy equilibrium agents can't go short in the security promising $r_{\mathbf{1}}$ and also can't go long in the security $\mathbb{1}$, income streams m that can be generated from the existing securities can not anymore be valued by a linear function. However Proposition 2 shows that we can give valuation bounds for income streams that can be replicated, similar as in the literature on portfolio constraints (see Luttmer [1996]).

4.4 Allocations: A Two Fund Separation Result

We can also characterize equilibrium allocations, such that the relationship to two fund separation theorems characteristic for the CAPM can be seen. Again in order to simplify notation and to make the analogy to the CAPM more visible, let us assume that short selling constraints on equity are not binding. The structure of bankruptcy equilibrium requires that we characterize the consumption-default plans of agents depending on whether they are long or short in the bond in a bankruptcy equilibrium. As in the case with pricing, the role of the aggregate endowment is now taken by the aggregate endowment corrected for aggregate shortfalls in promises $\tilde{\omega}_{\mathbf{1}}$. Since constraints on the possible bond positions (z_b^{i+}, z_b^{i-}) may bind for some agents, we get additional terms that depend on the Lagrangian multipliers of the respective constraints.

Proposition 3. *Let $(\bar{x}, \bar{z}, \bar{q}, \bar{r}_{\mathbf{1}})$ is a bankruptcy equilibrium of the economy $\mathcal{E}(u, \omega, \delta, V)$ with non-binding short selling constraints on equity and strictly positive date zero consumption $\bar{x}_0^i$: If in a bankruptcy equilibrium an agent i trades long in the bond, her equilibrium consumption plan is given by*

$$\bar{x}_{\mathbf{1}}^i = \omega_{\mathbf{1}}^i + P_{Y_{b+}}\left((\alpha_1^i - \frac{\alpha_0^i}{\alpha_0}\alpha_1)\mathbb{1} - (\omega_{\mathbf{1}}^i + Y\delta^i - \frac{\alpha_0^i}{\alpha_0}\tilde{\omega}_{\mathbf{1}})\right) - \sigma_{b+}\frac{\alpha_0^i}{\alpha_0} r_{\mathbf{1}e},$$

where $\sigma_{b+} := \sum_{i=1}^{I} \sigma_{b+}^i$ *is the sum of all agents' Lagrange multipliers corresponding to the constraints* $z_{b+}^i \geq 0$, $P_{Y_{b+}}$ *is the projection on the span of* $(r_{\mathbf{1}}, Y)$ *and*

$$r_{\mathbf{1}e} := \frac{r_{\mathbf{1}} - P_Y(r_{\mathbf{1}})}{||r_{\mathbf{1}} - P_Y(r_{\mathbf{1}})||^2}.$$

If agent i trades short in the bond, her equilibrium allocation is given by

$$\bar{x}_{\mathbf{1}}^i = \omega_{\mathbf{1}}^i + P_{Y_{b-}} \left((\alpha_1^i - \frac{\alpha_0^i}{\alpha_0}\alpha_1)\mathbb{1} - (\omega_{\mathbf{1}}^i + Y\delta^i - \frac{\alpha_0^i}{\alpha_0}\tilde{\omega}_{\mathbf{1}}) \right) - \sigma_{b-}\frac{\alpha_0^i}{\alpha_0}\mathbb{1}_e,$$

where $\sigma_{b-} := \sum_{i=1}^{I} \sigma_{b-}^i$ *is the sum of all agents' Lagrange multipliers corresponding to the constraints* $z_{b-}^i \geq 0$, $P_{Y_{b+}}$ *is the projection on the span of* $(-\mathbb{1}, Y)$ *and*

$$\mathbb{1}_e := \frac{\mathbb{1} - P_Y(\mathbb{1})}{||\mathbb{1} - P_Y(\mathbb{1})||^2}.$$

If in equilibrium agent i does not trade in the bond, her equilibrium allocation is given by

$$\bar{x}_{\mathbf{1}}^i = \omega_{\mathbf{1}}^i + P_Y \left((\alpha_1^i - \frac{\alpha_0^i}{\alpha_0}\alpha_1)\mathbb{1} - (\omega_{\mathbf{1}}^i + Y\delta^i - \frac{\alpha_0^i}{\alpha_0}\tilde{\omega}_{\mathbf{1}}) \right).$$

where P_Y *is the projection on the span of* Y.

Proof: The proof is given in the appendix. □

From Proposition 3 we see that in the bankruptcy equilibrium consumption is characterized by an "approximate" linear risk sharing rule both for agents long and short in the bond with the aggregate endowment corrected for shortfalls in the promises made by bond trading (compare to the no bankruptcy case in Magill and Quinzii [1997]).

In the standard CAPM an investor's date 1 consumption is obtained from aggregate income via a linear sharing rule. Therefore investors hold fully diversified portfolios and each investor's portfolio is proportional to the "market portfolio". In a bankruptcy equilibrium this property is changed in two ways. Firstly and most importantly, even if the aggregate endowment is in the span of the available equities and can be interpreted as an aggregate output, the relevant benchmark portfolio is endogenous. This is because aggregate output has to be corrected for the planned shortfall in financial promises. Second, the optimal date one consumption allocations and thus portfolio decisions differ between agents who are short in the bond and agents who are long in the bond. It is as if the agents who are going long in the bond and agents who are going short in the bond face a different asset structure. For the first class of agents the bond pays $r_{\mathbf{1}}$, for the others the relevant promise is $\mathbb{1}$.

Proposition 2 and 3 show the analogy between the standard CAPM and the bankruptcy CAPM. They do not allow to calculate equilibrium values explicitly from the exogenous parameters, since the short positions of the bond traders have to be known to determine $\tilde{\omega}_{\mathbf{1}}$, the shortfall corrected market portfolio. In contrast to other models in the literature, the modeling approach to bankruptcy suggested here allows us to arrive at characterizations of equilibrium prices and allocations which are more general than fully parameterized numerical examples. These results are useful for the further study of the economics of bankruptcy and default in a general equilibrium context.

In particular, the explicit equilibrium characterization presented above provides new insights into the question of the potential welfare benefits of default in incomplete markets, as suggested by the examples in Zame [1993] and Dubey et al. [2005]. In combination with the analysis in Magill and Quinzii [1997] it allows one to investigate these issues more systematically.

5 Endogenous Risk

Our model provides a framework for studying some recent developments in credit risk modeling (see McNeil et al. [2005] for an overview). Most models in this literature are applied in risk management at banks and financial institutions. Taking the probability of default, the recovery rate and the exposure to default as determined by an exogenous source of randomness, these models try to capture default risk and credit losses from debt exposures by the implied loss distribution of a portfolio of debt instruments. From this viewpoint, risk management decisions are treated as a game against nature or a single person decision problem. Though our model contains also exogenous randomness, modeled by different states of the world, important parameters of credit risk are determined endogenously by the self interested interactions of individuals. Hence, credit risk is a consequence of an equilibrium rather than independent decisions. Exploring the consequences for risk management if credit risk is endogenous is an interesting problem for future research. We see our paper as a useful starting point for such a project.

Viewing credit risk as an endogenous variable has also consequences for analyzing liquidity. Figure 1 in our introductory example makes it immediately clear that the liquidity of the bond market is directly related to the credit risk of the bond. If credit risk is high, return spreads between long and short bond positions are high. From this perspective, it makes no sense to think of liquidity as an exogenous property of certain asset classes, as one frequently observes it in policy debates. As our model suggests, liquidity, like credit risk, is endogenous and an equilibrium phenomenon.

6 Conclusions

In this paper, we studied an economy with credit where bankruptcy can occur. An equilibrium with bankruptcy is a simple generalization of the standard general equilibrium model with financial markets. The two key ideas of our model are (i) to extend the choices of agents to negative consumption plans, interpreting the utility of negative consumption as a bankruptcy penalty, and (ii) to propose a bankruptcy clearing mechanism which redistributes the remaining assets of bankrupt agents to the creditors. This mechanism generates an endogeneous rationally expected return rate on loans. Together with the assumptions that individuals hold strictly positive commodity endowments and can only default when their resources are exhausted, the bankruptcy mechanism also rules out equilibria where financial market trading breaks down because of extreme pessimism, a problem which plagues the literature on default in general equilibrium. Moreover, combining our bankruptcy equilibrium model with the assumption of mean variance preferences allows one to derive a bankruptcy version of the Capital Asset Pricing Model (CAPM).

Because of its tractability, we expect this bankruptcy model to be useful for economic applications. In particular, the CAPM version of the model provides an interesting starting point for studying asset pricing when bankruptcies may occur in some states. Two immediate questions, which we hope to pursue in the future, concern the welfare effects of bankruptcy and the effect of bankruptcy on credit risk management.

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7 Appendix

7.1 Proof of Proposition 1

Our proof consists of three steps:[16] In Step (i), we prove some preliminary results. In particular, we show that asset returns induce future state-contingent consumption which can be evaluated by an indirect utility function. Moreover, a consumer will not buy and sell bonds simultaneously, the lower bound on borrowing implies a bounded set of feasible asset trades, and the bankruptcy clearing mechanism implies a strictly positive return of the bond in all states. Step (ii) provides an existence proof for an asset market equilibrium with a bankruptcy clearing mechanism in period $t = 0$. For this proof, we can adapt standard arguments of general equilibrium theory. Finally, in Step (iii), we show that the bankruptcy clearing mechanism guarantees non-negative consumption in all states.

7.1.1 Preliminary results

The indirect utility function: Consider the price simplex in $\mathbb{R}_+^{J+2}$,

$$Q := \left\{ (p_0, q_b, q_e) \in \mathbb{R}_+^{J+2} | \; p_0 + q_b + q_e \mathbf{1}_J = 1 \right\}.$$

For given $r_{\mathbf{1}} \in (0,1]^S$ and $(p_0, q_b, q_e) \in Q$, each consumer $i \in I$ maximizes $u^i(x_0^i, x_{\mathbf{1}}^i)$ by choosing asset trades $(z_{b+}^i, z_{b-}^i, z_e^i)$ and a consumption allocation $(x_0^i, x_{\mathbf{1}}^i) \in X^i$ subject to[17]

$$\begin{array}{llcl}
(a) & p_0\left(x_0^i - \omega_0^i\right) & \leq & -q_b z_{b+}^i + q_b z_{b-}^i - q_e z_e^i, \\
(b) & x_{\mathbf{1}}^i - \omega_{\mathbf{1}}^i - Y\delta^i & \leq & r_{\mathbf{1}} z_{b+}^i - \mathbb{1} z_{b-}^i + Y z_e^i, \\
(c) & z_{b+}^i & \geq & 0 \\
(d) & z_{b-}^i & \geq & 0, \\
(e) & z_{b-}^i & \leq & \kappa, \\
(f) & z_e^i & \geq & -\delta^i,
\end{array}$$

where the constraint (e) follows from A5 in Proposition 1.

Since u^i is strictly increasing in its arguments (Proposition 1, A2), condition (b) is binding for an optimal choice $(x_0^i, x_{\mathbf{1}}^i, z_{b+}^i, z_{b-}^i, z_e^i)$, i.e.,

$$x_{\mathbf{1}}^i = \omega_{\mathbf{1}}^i + Y\left(\delta^i + z_e^i\right) + r_{\mathbf{1}} z_{b+}^i - \mathbb{1} z_{b-}^i. \tag{8}$$

Moreover, for $r_{\mathbf{1}} \neq \mathbb{1}$, the consumer will never take a short position $z_{b-}^i > 0$ and a long position $z_{b+}^i > 0$ simultaneously. These observations allow us to restrict attention to net trades in bonds $z_b^i := \left(z_{b+}^i - z_{b-}^i\right) = \left(z_b^i \vee 0\right) + \left(z_b^i \wedge 0\right) \in \mathbb{R}$. Hence, we can write $z^i = (z_b^i, z_e^i)$ for the net asset trades of a consumer $i \in I$.

For each consumer $i \in I$, consider the consumption set in period 0, $X_0^i := \mathbb{R}_+$, and the set of feasible net asset trades, $Z^i := [-\kappa, \infty) \times \left(\mathbb{R}_+^J - \left\{\delta^i\right\}\right)$.

[16] We would like to thank an anonymous referee of this journal for suggesting the greatly simplified proof of existence for an equilibrium in period $t = 0$.

[17] Notice the change of normalisation for the budget in $t = 0$.

For any $r_1 \in (0,1]^S$, equation (8) allows us to define the following *indirect utility function* $v^i : X_0^i \times Z^i \to \mathbb{R}$

$$v^i(x_0^i, z_b^i, z_e^i; r_\mathbf{1}) := u^i(x_0^i, \omega_\mathbf{1}^i + Y\left(\delta^i + z_e^i\right) + r_\mathbf{1}\left(z_b^i \vee 0\right) + \mathbb{1}\left(z_b^i \wedge 0\right))$$

By Assumption A2, the indirect utility function v^i is a *continuous* function. Moreover, v^i is *strictly increasing* and *concave* in $\left(x_0^i, z_b^i, z_e^i\right)$.

Feasible allocations: An allocation for a consumer $i \in I$ in period $t = 0$ is a vector of period 0 consumption and of asset trades $(x_0^i, z^i) \in X_0^i \times Z^i$. Denote by $A := \prod_{i \in I}\left(X_0^i \times Z^i\right) \subset \mathbb{R}^{J+2}$ the set of allocations for the economy. Denote $(x_0^i, z^i)_{i\in I}$ by (x_0, z). An allocation $(x_0, z) \in A$ is *weakly feasible* if $\sum_{i\in I}\left(x_0^i - \omega_0^i, z^i\right) \leq (0,0)$ holds. Since $X_0^i \times Z^i$ is bounded below for all $i \in I$, the set of weakly feasible allocations

$$F := \left\{(x_0^i, z^i)_{i\in I} \in A \mid \sum_{i\in I}\left(x_0^i - \omega_0^i, z^i\right) \leq (0,0)\right\}$$

is a compact set.

The bankruptcy clearing mechanism The *bankruptcy clearing mechanism* (compare Equation 4) is a function $\rho : A \to (0,1]^S$ defined by its component functions

$$\rho_s(x_0, z) = \begin{cases} \frac{\sum_{i=1}^I\left(-z_b^i \vee 0\right)\wedge(\omega_s^i + Y_s(\delta^i + z_e^i))}{\sum_{i=1}^I\left(-z_b^i \vee 0\right)}. & \text{for } \sum_{i=1}^I\left(-z_b^i \vee 0\right) > 0 \\ 1 & \text{for } \sum_{i=1}^I\left(-z_b^i \vee 0\right) = 0 \end{cases},$$

$\rho_s(x_0, z)$ is a continuous function on A. Notice that, by Assumption A3, $\omega_s^i > 0$ for all $s \in S$ and all $i \in I$. Hence, $\left(-z_b^i \vee 0\right) \wedge (\omega_s^i + Y_s(\delta^i + z_e^i)) = -z_b^i$ for $0 < -z_b^i < \omega_s^i$. This implies that $\rho_s(x_0, z) = 1$ in a neighborhood of $z_b = 0$. Thus, $\rho_s(x_0, z)$ is well-defined and continuous at $z_b = 0$. Moreover, $\frac{\left(-z_b^i \vee 0\right)\wedge(\omega_s^i + Y_s(\delta^i + z_e^i))}{\left(-z_b^i \vee 0\right)}$ is an increasing function of z_b^i with a minimum $\frac{\kappa \wedge (\omega_s^i + Y_s(\delta^i + z_e^i))}{\kappa} \geq \frac{\kappa \wedge \omega_s^i}{\kappa}$ at $z_b^i = -\kappa$. Therefore, we have

$$\begin{aligned}
\rho_s(x_0, z) &= \frac{\sum_{i=1}^I\left(-z_b^i \vee 0\right) \wedge (\omega_s^i + Y_s(\delta^i + z_e^i))}{\sum_{i=1}^I\left(-z_b^i \vee 0\right)} \\
&= \sum_{i=1}^I\left[\frac{\left(-z_b^i \vee 0\right) \wedge (\omega_s^i + Y_s(\delta^i + z_e^i))}{\left(-z_b^i \vee 0\right)}\right]\frac{\left(-z_b^i \vee 0\right)}{\sum_{i=1}^I\left(-z_b^i \vee 0\right)} \\
&\geq \sum_{i=1}^I\left[\frac{\kappa \wedge \omega_s^i}{\kappa}\right]\frac{\left(-z_b^i \vee 0\right)}{\sum_{i=1}^I\left(-z_b^i \vee 0\right)} \\
&\geq \min_{i\in I}\left[\frac{\kappa \wedge \omega_s^i}{\kappa}\right]\underbrace{\sum_{i=1}^I\frac{\left(-z_b^i \vee 0\right)}{\sum_{i=1}^I\left(-z_b^i \vee 0\right)}}_{=1} \\
&= \min_{i\in I}\left[\frac{\kappa \wedge \omega_s^i}{\kappa}\right] =: \underline{\rho}_s > 0.
\end{aligned}$$

Hence, for all $s \in S$, $\rho_s(x_0, z)$ is bounded below by a strictly positive value $\underline{\rho}_s$. Denote by $R := \prod_{s \in S} \left[\underline{\rho}_s, 1\right] \subset (0,1]^S$ the set of feasible return rates for the bond.

7.1.2 Existence of equilibrium in $\mathrm{t} = 0$

Choose a compact and convex set $K \subset \mathbb{R}^{J+2}$ such that $(x_0^i, z^i)_{i \in I} \in F$ implies $(x_0^i, z^i) \in \operatorname{int} K$ for all $i \in I$.

For any $(p_0, q_b, q_e) \in Q$, denote by

$$\mathbb{B}_c^i(p_0, q_b, q_e) := \left\{(x_0^i, z_b^i, z_e^i) \in \left(X_0^i \times Z^i\right) \cap K |\ p_0 \left(x_0^i - \omega_0^i\right) + q_b z_b^i + q_e z_e^i \leq 0\right\}$$

the *bounded budget correspondence* for $t = 0$ restricted to the compact set K.

Lemma 1. *$\mathbb{B}_c^i : Q \to \left(X_0^i \times Z^i\right) \cap K$, is a compact- , convex-valued, and continuous correspondence.*

Proof: Since $\mathbb{B}_c^i(p_0, q_b, q_e) \subset K$ for all $(p_0, q_b, q_e) \in Q$, the budget set is compact. It is obviously convex. Since $\omega_0^i > 0$, $\kappa > 0$ and $\delta^i \in \mathbb{R}_{++}^J$, continuity follows by standard arguments, e.g., Debreu [1956], p. 63. □

For $r_{\mathbf{1}} \in R$ and $(p_0, q_b, q_e) \in Q$, define the *bounded demand correspondence* $f_c^i : Q \times R \to \mathbb{R}^{J+2}$ by

$$f_c^i(p_0, q_b, q_e; r_{\mathbf{1}}) := \arg\max \left\{v^i(x_0^i, z_b^i, z_e^i; r_{\mathbf{1}}) |\ \left(x_0^i, z_b^i, z_e^i\right) \in \mathbb{B}_c^i(p_0, q_b, q_e)\right\}.$$

Lemma 2. *$f_c^i(p_0, q_b, q_e; r_{\mathbf{1}})$ is a non-empty, compact- and convex-valued, u.h.c. correspondence from $Q \times R$ into $\mathbb{R}^{J+2}$.*

Proof: The indirect utility v^i is a continuous function on $\mathbb{R}_+ \times \mathbb{R}^{J+1} \times R$. By Lemma 1, $\mathbb{B}_c^i(p_0, q_b, q_e)$ is a compact-, convex-valued and continuous correspondence on Q. Hence, by the maximum theorem, the demand correspondence $f_c^i(p_0, q_b, q_e; r_{\mathbf{1}})$ is non-empty, compact-valued, and u.h.c. for all $(p_0, q; r_{\mathbf{1}}) \in Q \times R$. Since v^i is concave it follows by standard arguments that $f_c^i(p_0, q_b, q_e; r_{\mathbf{1}})$ is convex-valued. □

Lemma 3. *Consider a vector of prices $(p_0, q_b, q_e) \in Q$ and an allocation of optimal choices $(x_0^i, z^i) \in f_c^i(p_0, q_b, q_e; r_{\mathbf{1}})$ for all $i \in I$ which is weakly feasible, $(x_0^i, z^i)_{i \in I} \in F$. Then all prices must be strictly positive, $(p_0, q_b, q_e) >> 0$.*

Proof: Suppose there was a price $q_e^j = 0$, since the indirect utility is strictly increasing in (x_0^i, z^i), $z_{e_j}^i$ must be in the boundary of K. Since $(x_0^i, z^i)_{i \in I} \in F$ implies $(x_0^i, z^i) \in \operatorname{int} K$ for all $i \in I$, this contradicts $(x_0^i, z^i)_{i \in I} \in F$. A similar argument holds for $p_0 = 0$ and $q_b = 0$. □

Lemma 3 shows also that a weakly feasible allocation of optimal choices cannot be in the boundary of K. Hence, for a feasible allocation, the constraints of K will never be binding.

Define the *individual excess demand correspondence* $\zeta^i : Q \times R \to \mathbb{R}^{J+2}$ as

$$\zeta^i(p_0, q; r_{\mathbf{1}}) := f_c^i(p_0, q; r_{\mathbf{1}}) - \left\{(\omega_0^i, 0, 0)\right\}.$$

Obviously, ζ^i inherits all relevant properties of f_c^i, in particular, ζ^i is a non-empty, compact- and convex-valued u.h.c. correspondence. Since $f_c^i(p_0, q; r_{\mathbf{1}}) \subseteq \mathbb{B}_c^i(p_0, q)$, individual excess demand correspondences ζ^i are bounded below by $(-\omega_0^i, -\kappa, -\delta^i)$. Consequently, the aggregate excess demand correspondence $\sum_{i \in I} \zeta^i(p_0, q; r_{\mathbf{1}})$ is bounded below by $\left(-\sum_{i \in I} \omega_0^i, -|I|\kappa, -\sum_{i \in I} \delta^i\right)$.

We denote by $\zeta : Q \times R \to K^I$,

$$\zeta(p_0, q; r_{\mathbf{1}}) := \prod\nolimits_{i \in I} \zeta^i(p_0, q; r_{\mathbf{1}}),$$

the Cartesian product of the individual excess demand correspondences. By Lemma 2, $\zeta(p_0, q; r_{\mathbf{1}})$ is a non-empty, compact- and convex-valued u.h.c. correspondences on $Q \times R$ as a product of correspondences with these properties.

For any excess demand vector $(x_0, z) := (x_0^i, z^i)_{i \in I} \in K^{|I|}$, let

$$\mu(x_0, z) := \arg\max \left\{ p_0 \sum_{i=1}^{I} \left(x_0^i - \omega_0^i\right) + q_b \sum_{i=1}^{I} z_b^i + q_e \sum_{i=1}^{I} z_e^i | \; (p_0, q_b, q_e) \in Q \right\}$$

be the set of prices in Q that maximize the value of the excess demand. Since Q is compact and $p_0 \sum_{i=1}^{I} \left(x_0^i - \omega_0^i\right) + q_b \sum_{i=1}^{I} z_b^i + q_e \sum_{i=1}^{I} z_e^i$ is a continuous function on Q, $\mu(x_0, z)$ is not empty. By the maximum theorem, $\mu : K^{|I|} \to Q$ is a compact, convex-valued, and u.h.c. correspondence on $K^{|I|}$.

For each $(p_0, q_b, q_e, r_{\mathbf{1}}, (x_0, z)) \in Q \times R \times K^{|I|}$, define the correspondence

$$\Phi(p_0, q_b, q_e, r_{\mathbf{1}}, (x_0, z)) := \mu(x_0, z) \times \{\rho(x_0, z)\} \times \zeta(p_0, q_b, q_e, r_{\mathbf{1}}).$$

The correspondence Φ is a mapping $Q \times R \times K^{|I|} \to Q \times R \times K^{|I|}$. The correspondence Φ is non-empty, compact-valued, convex-valued, and u.h.c., as a Cartesian product of correspondences with these properties. Hence, by Kakutani's fixed point theorem, there is $(p_0^*, q_b^*, q_e^*, r^*{}_{\mathbf{1}}, (x_0^*, z^*)) \in \Phi(p_0^*, q_b^*, q_e^*, r_{\mathbf{1}}^*, (x_0^*, z^*))$.

By construction of μ,

$$p_0^* \sum_{i=1}^{I} \left(x_0^{*i} - \omega_0^i\right) + q_b^* \sum_{i=1}^{I} z_b^{*i} + q_e^* \sum_{i=1}^{I} z_e^{*i} \geq p_0 \sum_{i=1}^{I} \left(x_0^{*i} - \omega_0^i\right) + q_b \sum_{i=1}^{I} z_b^{*i} + q_e \sum_{i=1}^{I} z_e^{*i}$$

for all $(p_0, q_b, q_e) \in Q$. Moreover, by the budget constraint,

$$p_0^* \left(x_0^{*i} - \omega_0^i\right) + q_b^* z_b^{*i} + q_e^* z_e^{*i} \leq 0$$

for all $i \in I$. Hence,

$$p_0 \sum_{i=1}^{I} \left(x_0^{*i} - \omega_0^i\right) + q_b \sum_{i=1}^{I} z_b^{*i} + q_e \sum_{i=1}^{I} z_e^{*i} \leq 0$$

for all $(p_0, q_b, q_e) \in Q$. This implies

$$\sum_{i \in I} \left(x_0^{*i} - \omega_0^i\right) \leq 0, \; \sum_{i \in I} z^{*i} \leq 0.$$

Therefore, the allocation $(x_0^{*i}, z^{*i})_{i\in I}$ is weakly feasible, i.e., $(x_0^*, z^*) \in F$.

Since $(x_0^{*i}, z^{*i}) \in f_c^i(p_0^*, q_b^*, q_e^*, r_{\mathbf{1}}^*)$ for all $i \in I$ and $(x_0^{*i}, z^{*i})_{i\in I}$ is a weakly feasible allocation, it follows from Lemma 3 that all prices must be strictly positive, $(p_0^*, q_b^*, q_e^*) >> 0$. Moreover, since the indirect utility is strictly increasing the budget constraint must be binding for all $i \in I$, i.e., $p_0^* \left(x_0^{*i} - \omega_0^i\right) + q_b^* z_b^{*i} + q_e^* z_e^{*i} = 0$. Together with weak feasibility, this implies market clearing,

$$\sum_{i\in I} \left(x_0^{*i} - \omega_0^i\right) = 0, \ \sum_{i\in I} z^{*i} = 0.$$

7.1.3 Existence of equilibrium in t=1

It remains to show that in a bankruptcy equilibrium

(i) $r_{\mathbf{1}}^* = \rho(x_0^*, z^*)$ and

(ii) $(x_0^{*i}, z^{*i}) \in f_c^i(p_0^*, q_b^*, q_e^*, r_{\mathbf{1}}^*)$ for all $i \in I$ with $\sum_{i\in I} z^{*i} = 0$,

consumption will be non-negative in every state $s \in S$, i.e., condition (iii) of the definition of a bankruptcy equilibrium

$$\sum_{i=1}^{I} \left(x_s^{*i} \vee 0\right) = \sum_{i=1}^{I} (\omega_s^i + Y_s \delta^i)$$

for all $s \in S$ is also satisfied.

Recall that, for an equilibrium price system $(p_0^*, q_b^*, q_e^*, r_{\mathbf{1}}^*)$ and an equilibrium allocation $(x_0^{*i}, z^{*i}) \in f_c^i(p_0^*, q_b^*, q_e^*, r_{\mathbf{1}}^*)$, by Equation 8,

$$x_s^{*i} = \omega_s^i + Y_s \left(\delta^i + z_e^{*i}\right) + r_s (z_b^{*i} \vee 0) + (z_b^{*i} \wedge 0)$$

and, by Equation 4,

$$r_{\mathbf{1}}^* = \rho(x_0^*, z^*) := \frac{\sum_{i=1}^{I} \left(-z_b^{*i} \vee 0\right) \wedge (\omega_s^i + Y_s(\delta^i + z_e^{*i}))}{\sum_{i=1}^{I} (-z_b^{*i} \vee 0)}$$

for all $s \in S$ hold. These conditions imply non-negative consumption in all states.

Lemma 4. *If for all $i \in I$ and all $s \in S$ equations 8 and 4 hold, then*

$$\textstyle\sum_{i=1}^{I} z_b^{*i} = 0 \quad \text{ and } \quad \sum_{i=1}^{I} z_e^{*i} = 0$$

imply

$$\sum_{i=1}^{I} \left(x_s^{*i} \vee 0\right) = \sum_{i=1}^{I} (\omega_s^i + Y_s \delta^i)$$

for all $s \in S$.

Proof: For notational convenience, we drop the reference * to the equilibrium values. Consider an arbitrary $s \in S$. By Equation 8, summing over all consumers $i \in I$, one obtains

$$\sum_{i=1}^{I} \left(x_s^i - \omega_s^i - Y_s\delta^i\right) = \sum_{i=1}^{I} \left[r_s(z_b^i \vee 0) + (z_b^i \wedge 0) + Y_s z_e^i\right],$$

which is equivalent to

$$\sum_{i=1}^{I} \left[(x_s^i \vee 0) - \omega_s^i - Y_s\delta^i\right] = \sum_{i=1}^{I} \left[r_s(z_b^i \vee 0) + (z_b^i \wedge 0) + Y_s z_e^i - (0 \wedge x_s^i)\right].$$

The asset market equilibrium conditions

$$\textstyle\sum_{i=1}^{I} z_b^i = 0 \quad \text{and} \quad \sum_{i=1}^{I} z_e^i = 0$$

imply

$$\begin{aligned}
& \sum_{i=1}^{I} \left[r_s(z_b^i \vee 0) + (z_b^i \wedge 0) + Y_s z_e^i - (x_s^i \wedge 0)\right] \\
= \;& r_s \underbrace{\sum_{i=1}^{I}(z_b^i \vee 0)}_{=-\sum_{i=1}^{I}(z_b^i \wedge 0)} + \sum_{i=1}^{I}(z_b^i \wedge 0) - \sum_{i=1}^{I}(x_s^i \wedge 0) + Y_s \underbrace{\sum_{i=1}^{I} z_e^i}_{=0} \\
= \;& r_s \sum_{i=1}^{I}(-z_b^i \vee 0) + \sum_{i=1}^{I}(z_b^i \wedge 0) - \sum_{i=1}^{I}(x_s^i \wedge 0).
\end{aligned}$$

Denote by $I^+ := \{i \in I|\ z_b^i \geq 0\}$ the set of consumers who are not borrowing, by $I_s^- := \{i \in I|\ z_b^i < 0,\ x_s^i \geq 0\}$ the set of loan takers who are solvent, and by $I_i^- : \{i \in I|\ z_b^i < 0,\ x_s^i < 0\}$ consumers who are insolvent, then substituting Equation 4 for r_s yields

$$\begin{aligned}
& r_s \sum_{i=1}^{I}(-z_b^i \vee 0) + \sum_{i=1}^{I}(z_b^i \wedge 0) - \sum_{i=1}^{I}(x_s^i \wedge 0) \\
= \;& r_s \sum_{i \in I^-}(-z_b^i \vee 0) + \sum_{i \in I^-}(z_b^i \wedge 0) - \sum_{i \in I^-}(x_s^i \wedge 0) \\
= \;& \sum_{i \in I^-} \left[\left(-z_b^i\right) \wedge \left(\omega_s^i + Y_s(\delta^i + z_e^i)\right)\right] - \sum_{i \in I^-}(-z_b^i) - \sum_{i \in I^-}(x_s^i \wedge 0) \\
= \;& \sum_{i \in I_i^-} \left[\omega_s^i + Y_s(\delta^i + z_e^i))\right] - \sum_{i \in I_i^-}(-z_b^i) - \sum_{i \in I_i^-}(x_s^i) \\
= \;& \sum_{i \in I_i^-} \left[\left(\omega_s^i + Y_s\left(\delta^i + z_e^i\right) + z_b^i\right) - x_s^i)\right] = 0,
\end{aligned}$$

since $x_s^i < 0$ implies $x_s^i = \omega_s^i + Y_s\left(\delta^i + z_e^i\right) + z_b^i < 0$. Hence, $\sum_{i=1}^{I}\left(x_s^i \vee 0\right) = \sum_{i=1}^{I}(\omega_s^i + Y_s\delta^i)$. $\square$

7.2 Proof of Proposition 2

We partition the matrix

$$T = \begin{bmatrix} -q_b & q_b & -q_e \\ r_{\mathbf{1}} & -\mathbb{1} & Y \end{bmatrix}$$

and the portfolio vector $z^i \in Z^i$ into long-bond, short-bond, and equity trades by $T = (T_{b+}, T_{b-}, T_e)$ and $z^i = (z^i_{b+}, z^i_{b-}, z^i_e)^T$, respectively. Using Lagrange multipliers $\pi^i \in \mathbb{R}^{S+1}$, $\sigma^i_{b+} \geq 0$, and $\sigma^i_{b-} \geq 0$, and assuming we are at an interior solution (δ^i are such that short sales constraints on equity are not binding at equilibrium, and parameters are such that $x^i_0 > 0$ at equilibrium), the KKT conditions for the minimization of the Lagrange function

$$L^i(x^i, z^i, \pi^i, \sigma^i_{b+}, \sigma^i_{b-}) = -u^i(x^i) + \langle \pi^i, x^i - \omega^i - e^i - Tz^i \rangle - \sigma^i_{b+} z^i_{b+} - \sigma^i_{b-} z^i_{b-}$$

imply that

$$\begin{aligned} \langle \nabla u^i(\bar{x}^i), T_e \rangle &= (0, \ldots, 0), \\ \langle \nabla u^i(\bar{x}^i), T_{b+} \rangle &= -\sigma^i_{b+} \leq 0, \text{ and} \\ \langle \nabla u^i(\bar{x}^i), T_{b-} \rangle &= -\sigma^i_{b-} \leq 0, \end{aligned}$$

Since the optimization problem is convex, the KKT conditions are also sufficient for an optimal solution to the agent's problem.

The gradient of the linear quadratic utility function fulfills in the equilibrium allocation $\bar{x}^i$

$$\langle \nabla u^i(\bar{x}^i), \tau \rangle \leq 0 \quad \forall \tau \in \mathcal{C}.$$

Summing up all agent's equilibrium gradients we define the vector

$$\bar{\gamma} := \sum_{i=1}^{I} \nabla u^i(\bar{x}^i) = (\alpha_0, \alpha_1 \mathbb{1} - ((\omega_{\mathbf{1}} + Y\delta) - d_{\mathbf{1}}))^T,$$

where $d_{\mathbf{1}} = \sum_{i=1}^{I} d^i_{\mathbf{1}} := \sum_{i=1}^{I} (\mathbb{1} - r_{\mathbf{1}}) z^i_{b-}$ is the aggregate shortfall from promises on the bond. Still we have $\langle \bar{\gamma}, \tau \rangle \leq 0 \quad \forall \tau \in \mathcal{C}$. Since any trade $\tau \in \mathcal{C}$ can be decomposed as $\tau = (-c(m), m)$ we get for $\frac{1}{\alpha_0}\bar{\gamma}$ that $\frac{\bar{\gamma}_{\mathbf{1}}}{\alpha_0} = \frac{\alpha_1}{\alpha_0}\mathbb{1} - \frac{1}{\alpha_0}\tilde{\omega}_{\mathbf{1}}$ and $c(m) \geq \left\langle \frac{\bar{\gamma}_{\mathbf{1}}}{\alpha_0}, m \right\rangle$, which proves the proposition. $\square$

7.3 Proof of Proposition 3

Suppose agent i goes long in the bond. Define the matrix by $T_{\text{long}} = \begin{pmatrix} -q_b & q_e \\ r_{\mathbf{1}} & Y \end{pmatrix}$. We know from proposition 2 that

$$\langle T^T_{\text{long}}, \nabla u^i(\bar{x}^i) \rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and} \tag{9}$$

$$\langle T^T_{\text{long}}, \bar{\gamma} \rangle = \begin{pmatrix} -\sigma_{b+} \\ 0 \end{pmatrix}, \tag{10}$$

where $\nabla u^i(x^i) = (\alpha_0^i, \alpha_1^i - x_{\mathbf{1}}^i)^T$ and $\bar{\gamma} = (\alpha_0, \alpha_1 \mathbb{1} - \tilde{\omega}_{\mathbf{1}})^T$. Divide equation (9) by α_0^i and equation (10) by α_0 , subtract the equations and multiply the result by α_0^i again. With $\bar{\tau}_{\mathbf{1}}^i := \bar{x}_{\mathbf{1}}^i - \omega_{\mathbf{1}}^i - Y\delta^i$ this gives

$$\langle Y_{b+}^T, \bar{\tau}_{\mathbf{1}}^i \rangle = \langle Y_{b+}^T, (\alpha_1^i - \frac{\alpha_0^i}{\alpha_0}\alpha_1)\mathbb{1} - ((\omega_{\mathbf{1}}^i + Y\delta^i) - \frac{\alpha_0^i}{\alpha_0}\tilde{\omega}_{\mathbf{1}}) \rangle - \frac{\alpha_0^i}{\alpha_0} \begin{pmatrix} \sigma_{b+} \\ 0 \end{pmatrix}.$$

We can write

$$\begin{pmatrix} \sigma_{b+} \\ 0 \end{pmatrix} = \langle Y_{b+}^T, \sigma_{b+} \frac{r_{\mathbf{1}} - P_Y(r_{\mathbf{1}})}{||r_{\mathbf{1}} - P_Y(r_{\mathbf{1}})||^2} \rangle = \langle Y_{b+}^T, \sigma_{b+} r_{\mathbf{1}e} \rangle.$$

Now, since $\langle Y_{b+}^T, v_{\mathbf{1}} \rangle = \langle Y_{b+}^T, P_{Y_{b+}}(v_{\mathbf{1}}) \rangle$ for any vector $v_{\mathbf{1}} \in \mathbb{R}^S$, it follows that

$$\bar{\tau}_{\mathbf{1}}^i = P_{Y_{b+}} \left((\alpha_1^i - \frac{\alpha_0^i}{\alpha_0}\alpha_1)\mathbb{1} - ((\omega_{\mathbf{1}}^i + Y\delta^i) - \frac{\alpha_0^i}{\alpha_0}\tilde{\omega}_{\mathbf{1}}) - \sigma_{b+} \frac{\alpha_0^i}{\alpha_0} r_{\mathbf{1}e} \right).$$

Finally, since $r_{\mathbf{1}} - P_Y(r_{\mathbf{1}}) \in \mathrm{span}(Y_{b+})$, the result follows.

The results for agents going short and for agents that do not trade in the bond are proved similarly. □